[redacted]

THE ANTENNA LABORATORY

TAP 1/9/65

RESEARCH ACTIVITIES in ---

Automatic Controls	*Antennas*	*Echo Area Studies*
Microwave Circuits	*Astronautics*	*E M Field Theory*
Terrain Investigations	*Radomes*	*Systems Analysis*
Wave Propagation		*Submillimeter Applications*

SUMMARY OF CONCEPTS AND TRANSFORMATIONS COMMONLY USED IN THE MATRIX DESCRIPTION OF POLARIZED WAVES

D. E. Barrick

Grant Number NsG-213-61

1388-16 9 April 1965

Prepared for:
National Aeronautics and Space Administration
Office of Grants and Research Contracts
Washington, D.C. 20546

Department of ELECTRICAL ENGINEERING

THE OHIO STATE UNIVERSITY
RESEARCH FOUNDATION
Columbus, Ohio

NOTICES

REPORT 1388-16

REPORT
by
THE OHIO STATE UNIVERSITY RESEARCH FOUNDATION
COLUMBUS, OHIO 43212

Sponsor	National Aeronautics and Space Administration Office of Grants and Research Contracts Washington, D.C. 20546
Grant Number	NsG-213-61
Investigation of	Theoretical and Experimental Analysis of the Electromagnetic Scattering and Radiative Properties of Terrain, with Emphasis on Lunar-like Surfaces
Subject of Report	Summary of Concepts and Transformations Commonly used in the Matrix Description of Polarized Waves
Submitted by	D.E. Barrick Department of Electrical Engineering Antenna Laboratory
Date	9 April 1965

ABSTRACT

2375 |

In this report, several commonly used matrices and transformations employed in the description of polarized waves and in the reception, scattering, and transmission of polarized radiation are found. The purpose is twofold: (1) To provide a complete development of these important concepts to the person unfamiliar with them, and (2) to provide a summary and reference of these matrices and their interdependence relationships to the person who must frequently employ them in analysis.

The first section discusses and defines the representation of general elliptically polarized waves in terms of orthogonal polarization states; two sucn sets of states are discussed in particular, the linear or Cartesian and the circular.

The second section develops and discusses the elementary scattering matrix for a surface, relating the scattered electric field to the incident electric field. Transformations between tne scattering matrix in linear polarization states and that in the circular states, and tne relationships between the circular scattering cross-sections and the linear scattering cross-sections are given in detail. The special case of back-scattering is also discussed.

The concept of the vector height of an antenna is applied to transmission and reception of polarized waves, and used to discuss the complete process of transmission, scattering, and reception of radiation. The determination of the elements of the scattering matrix of a stationary surface from measurements of power is discussed, and typical plots of power received from an arbitrary surface in various polarizations are shown.

The Stokes parameters of a polarized wave are developed in terms of the operational concept of the power received from an antenna. The matrix relating the Stokes parameters of a wave scattered from a surface to The Stokes parameters of the wave incident upon it is then derived in terms of the elements of the simple scattering matrix. Detailed tables are provided for a number of special cases of interest.

TABLE OF CONTENTS

		Page
I.	REPRESENTATION OF ELLIPTICALLY POLARIZED WAVES	1
II.	THE ELEMENTARY SCATTERING MATRIX AND SCATTERING CROSS-SECTION DEFINED	8
III.	ANTENNA VECTOR HEIGHT METHOD FOR SPECIFICATION OF RECEPTION AND TRANSMISSION OF POLARIZED WAVES	19
IV.	TRANSMISSION, SCATTERING, AND RECEPTION OF ELLIPTICALLY POLARIZED WAVES	23
V.	INTRODUCTION OF THE STOKES PARAMETERS	37
VI.	THE SCATTERING MATRIX FOR THE STOKES PARAMETERS	44
	REFERENCES	54

SUMMARY OF CONCEPTS AND TRANSFORMATIONS COMMONLY USED IN THE MATRIX DESCRIPTION OF POLARIZED WAVES

I. REPRESENTATION OF ELLIPTICALLY POLARIZED WAVES

Any general elliptically polarized electromagnetic wave the electric field of which is in the plane perpendicular to the direction of propagation may be represented in any number of ways[7, 9]. The most common are the use of two orthogonal unit vectors (either orthogonal in a real sense or in a Hermitian sense) to represent the polarized electric field. In all of these representations, the contravariant components along these two unit vectors (the electric field components) must be complex.

The most common representation of elliptical polarization is the use of two of the tnree Cartesian coordinates (the third, z here, being assumed as the direction of propagation). A wave with this representation has the following form:

$$\overline{E}(t) = E_x \hat{x} e^{j(\omega t - kz)} + E_y \hat{y} e^{j(\omega t - kz)} .$$

In this case, E_x and E_y are complex quantities, representable as shown below:

$$E_x = |E_x| e^{j\alpha}; \quad E_y = |E_y| e^{j\beta} .$$

However, one of the phase angles may be set equal to zero (or absorbed by a shift of $\omega t = \alpha$ in the specification of the time origin). Therefore write tnese components as

$$E_x = |E_x| ; \quad E_y = |E_y| e^{j\delta} ; \quad \delta = \beta - \alpha .$$

Here δ is the difference in phase between the x component of the E field and the y component. Now there are essentially three independent quantities which specify the polarization in the Cartesian system, two of these being the magnitudes (always defined positively) of the waves along the coordinate axes, and the third being the phase difference between the two.

Since it is true that three quantities, in general, are necessary to specify elliptical polarization in a Cartesian system of representation, then it is reasonable to assume that any system employing two orthogonal vectors to represent elliptical polarization will require three independent quantities to represent the wave completely.

Although it has not been mentioned, it is possible that these three independent quantities specifying the polarization (i.e., $|E_x|$, $|E_y|$, and δ) in the Cartesian system can vary with time. However, the generally accepted definition of "complete polarization" demands the following: (1) The ratio of the magnitudes must remain constant, and (2) the phase difference, δ, must remain constant, and may not change in sign. These requirements are understandable when polarization is viewed as a normalized ellipse traced out by the tip of the normalized field vector in the plane perpendicular to propagation. In order for this ellipse to maintain the same shape and orientation, it is necessary for the axial ratio to remain constant (contained in the ratio of electric field magnitudes), and also for the orientation of the ellipse and direction of motion of the tip of the field vector along the ellipse to remain unchanged (both determined from the sign and magnitude of the phase difference δ). Any wave which does not meet the above requirements of the definition for "complete polarization" is said to be "partially polarized", and may be represented as the sum of a completely polarized wave and a "randomly polarized" or "non-polarized" wave[11]. The discussion in this report is confined to completely polarized waves.

Many times it is convenient or necessary to resolve an elliptically polarized wave already specified in the x-y plane of one Cartesian system along the axes x'-y' of another Cartesian system, both with the z and z' axes coinciding with the direction of propagation. The relationship between these two Cartesian systems may be viewed as a rotation of the x'-y' axes of the second system by an angle ϕ in the counterclockwise sense about the z axis from the original x-y axes when looking in the direction of tne negative z axis, i.e., the direction from which the wave is coming. The components in the new x'-y' system are given by (see Fig. 1)

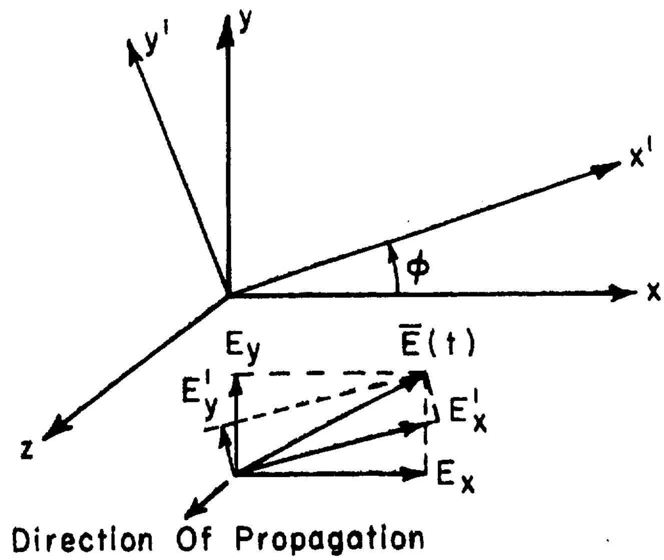

Fig. 1. Rotation of axes.

$$\overline{E}(t) = E_{x'}\hat{x}'\, e^{j(\omega t-kz)} + E_{y'}\hat{y}'\, e^{j(\omega t-kz)},$$

where

$$\hat{x}\cdot\hat{x}' = \hat{y}\cdot\hat{y}' = \cos\phi \; ; \quad -\hat{y}\cdot\hat{x}' = \sin\phi = \hat{x}\cdot\hat{y}' ,$$

and

$$E_{x'} = E_x \cos\phi + E_y \sin\phi$$

$$E_y' = -E_x \sin\phi + E_y \cos\phi .$$

Matrix representation gives a convenient way of expressing the above relation between the components in the x'-y' system and those in the original x-y system; matrix algebra can then be used to provide an orderly method of finding the inverse relationship, i.e., the components in the x-y system expressed in terms of the components in the x'-y' system. These representations are shown below.

$$\begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \; ; \; \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} .$$

Define

(1) $$[T_r] = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \; ; \; \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} = [T_r] \begin{bmatrix} E_x \\ E_y \end{bmatrix} .$$

Then

$$[T_r]^{-1} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \; ; \; \begin{bmatrix} E_x \\ E_y \end{bmatrix} = [T_r]^{-1} \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} .$$

One reason a person may have for resolving the components in one Cartesian system along newly defined axes x'-y' is that of making the new x'-y' axes lie along the major axes of the polarization ellipse. This is easily done by adjusting ϕ until the complex phase difference between the new set of components $E_{x'}$ and $E_{y'}$ is identically $\pi/2$.

Another convenient set of orthogonal unit vectors (this time orthogonal in an Hermitian sense) which proves quite useful in the representation of elliptically polarized waves are those which describe a right and left circularly polarized wave; such a representation is convenient because right and left circularly polarized waves are employed in many applications, and they are therefore familiar to many. In this system of representation, an elliptically polarized wave is specified as follows:

$$\overline{E}(t) = E_R \hat{r} e^{j(\omega t - kz)} + E_L \hat{\ell} e^{j(\omega t - kz)} .$$

As in the case of the Cartesian representation, E_R, and E_L are complex in general. The unit vectors, $\hat{r}$ and $\hat{\ell}$, associated with the right and left circularly polarized components, respectively, have the following properties:

$$(\hat{r}, \hat{\ell}) = \hat{r} \cdot \hat{\ell}^* = \hat{\ell} \cdot \hat{r}^* = (\hat{\ell}, \hat{r}) = 0$$

and

$$(\hat{r}, \hat{r}) = \hat{r} \cdot \hat{r}^* = (\hat{\ell}, \hat{\ell}) = \hat{\ell} \cdot \hat{\ell}^* = 1 \; .$$

The parenthetical notation above refers to the inner product in a Hermitian sense, and may be used interchangeably with the dot product notation as shown above.

In order to find the transformation between the components of an elliptically polarized wave expressed in The Cartesian system and its components in the circular system, it is helpful to examine first the expression for a right circular and left circular wave in the Cartesian system. Such an expression for the right circularly polarized wave is given below:

$$\overline{E}_R(t) = E_1 \hat{x} \, e^{j(\omega t - kz)} - jE_1 \hat{y} \, e^{j(\omega t - kz)} = \left(\frac{\hat{x} - j\hat{y}}{\sqrt{2}}\right) \sqrt{2} \, E_1 \; e^{j(\omega t - kz)} .$$

The expression for the left circularly polarized wave has the following form:

$$\overline{E}_L(t) = E_2 \hat{x} \, e^{j(\omega t - kz)} + jE_2 \hat{y} \, e^{j(\omega t = kz)} = \left(\frac{\hat{x} + j\hat{y}}{\sqrt{2}}\right) \sqrt{2} \, E_2 \, e^{j(\omega t - kz)} .$$

These expressions conform to the generally accepted definition of a right circularly polarized wave as one whose total electric field vector is rotating in a clockwise sense when looking along the direction of propagation; the left circularly polarized wave has a counter clockwise sense of rotation when viewed in the same direction.

It is evident immediately that the following inner products hold true:

$$\left(\frac{\hat{x} - j\hat{y}}{\sqrt{2}}\right) \cdot \left(\frac{\hat{x} - j\hat{y}}{\sqrt{2}}\right)^* = 1 = \left(\frac{\hat{x} + j\hat{y}}{\sqrt{2}}\right) \cdot \left(\frac{\hat{x} + j\hat{y}}{\sqrt{2}}\right)^*$$

and

$$\left(\frac{\hat{x} - j\hat{y}}{\sqrt{2}}\right) \cdot \left(\frac{\hat{x} + j\hat{y}}{\sqrt{2}}\right)^* = 0 = \left(\frac{\hat{x} + j\hat{y}}{\sqrt{2}}\right) \cdot \left(\frac{\hat{x} - j\hat{y}}{\sqrt{2}}\right)^* .$$

This suggests the logical definition for the unit vectors, $\hat{r}$ and $\hat{\ell}$, discussed previously, since both these sets of unit vectors obey the same inner product laws in a Hermitian sense; therefore, define

$$\hat{r} = \left(\frac{\hat{x}-j\hat{y}}{\sqrt{2}}\right) \quad \text{and} \quad \hat{\ell} = \left(\frac{\hat{x}+j\hat{y}}{\sqrt{2}}\right) \quad .$$

It will be desirable, then, to find a transformation which will relate the Cartesian components of a general elliptically polarized wave to the components in the circular system of representation. This transformation will be evident upon re-arranging the equation for an elliptical wave in Cartesian coordinates as follows:

$$\overline{E}(t) = E_x\hat{x}\, e^{j(\omega t-kz)} + E_y\hat{y}\, e^{j(\omega t-kz)} = [\, E_x\hat{x} + E_y\hat{y}]\;\; e^{j(\omega t-kz)}$$

or

$$\overline{E}(t) = \left[\frac{(E_x+jE_y)}{2}\hat{x} - j\frac{(E_x+jE_y)}{2}\hat{y} + \frac{(E_x-jE_y)}{2}\hat{x} + j\frac{(E_x-jE_y)}{2}\hat{y}\right] e^{j(\omega t-kz)};$$

$$\therefore \quad \overline{E}(t) = \left(\frac{E_x+jE_y}{\sqrt{2}}\right)\left(\frac{\hat{x}-j\hat{y}}{\sqrt{2}}\right)e^{j(\omega t-kz)} + \left(\frac{E_x-jE_y}{\sqrt{2}}\right)\left(\frac{\hat{x}+j\hat{y}}{\sqrt{2}}\right)e^{j(\omega t-kz)}$$

or

$$\overline{E}(t) = \left(\frac{E_x+jE_y}{\sqrt{2}}\right)\hat{r}\, e^{j(\omega t-kz)} + \left(\frac{E_x-jE_y}{\sqrt{2}}\right)\;\hat{\ell}\, e^{j(\omega t-kz)} \,.$$

From the last equation, the components of the electric field in the circular representation are obvious:

$$E_R = \frac{E_x+jE_y}{\sqrt{2}} \quad ; \quad E_L = \frac{E_x-jE_y}{\sqrt{2}} \quad .$$

Again matrix representation provides a convenient expression of the transformation between an elliptically polarized wave represented in the Cartesian system and the same wave represented in the circular system; matrix algebra again provides a simple means of finding the inverse transformation, i.e., the expression of the Cartesian components of an elliptically polarized wave in terms of the circular components. These transformations are given below.

$$\begin{bmatrix} E_R \\ E_L \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} \quad ; \quad \begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} E_R \\ E_L \end{bmatrix} .$$

Define

$$(2) \qquad [T_c] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \end{bmatrix} \quad ; \quad \begin{bmatrix} E_R \\ E_L \end{bmatrix} = [T_c] \begin{bmatrix} E_x \\ E_y \end{bmatrix} .$$

Then

$$[T_c]^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \quad ; \quad \begin{bmatrix} E_x \\ E_y \end{bmatrix} = [T_c]^{-1} \begin{bmatrix} E_R \\ E_L \end{bmatrix} .$$

As mentioned previously, the Cartesian and the circular representations of an elliptically polarized wave are only two of many methods using orthogonal unit vectors for representing the same wave. The representation in Cartesian coordinates is oftentimes alternatively referred to as representation by linear polarization states. It should be mentioned in passing that all these methods of representing an elliptically polarized wave by orthogonal unit vectors have one point in common; i.e., the components of the unit vectors in any orthogonal representation lie at opposite ends of a diameter through the Poincare' sphere. The two components in the Cartesian system lie at opposite ends of a diameter on the equator of the sphere. The two components in the circular system lie at the north and south pole of the sphere. Since there is an infinity of different possible diameters passing through this sphere, there is an infinity of possible orthogonal representations for any general elliptically polarized wave. Actually, any two different points on this sphere, whether they lie at opposite ends of a diameter or not, represent two different and independent polarization states (although not orthogonal), and they may be used to represent any elliptical wave (so long as neither of these points is a "null" point [6] for that particular wave).

It should also be noted that an elliptically polarized wave, toward which the full attention of this section has been devoted, is the most general case; linear and circular polarication of a wave are merely degenerate cases of elliptical polarization. Thus the study of the most general case applies equally well to these special degenerate cases.

The main points of this section have been a discussion of the number of necessary and sufficient independent pieces of information required for the complete specification of the polarization state of a wave, the definition of "complete polarization" of a wave, and a discussion of the method of representing a general elliptically polarized wave by orthogonal unit vectors was discussed. As one example, the Cartesian system was examined and the transformation from one Cartesian system, x-y, to another, x'-y', was found and discussed; the transformation and its inverse is a circular system; the transformation between the Cartesian and the circular system was derived, along with the inverse transformation, and is shown in Eq. (2).

II. THE ELEMENTARY SCATTERING MATRIX AND SCATTERING CROSS-SECTIONS DEFINED

An incoming electromagnetic wave is scattered by an element of surface dA, as shown in Fig. 2. In the most general case both the incident and the scattered fields are elliptically polarized, but do not necessarily have the same polarization; thus depolarization takes place because the surface element may involve some roughness, and because scattering is not restricted to the specular direction in general. The change in polarization state between incident and scattered wave upon striking a surface element involving roughness will also be a function of the frequency (and therefore wavelength of the wave, and also upon the orientation of the element of area dA.

Since any two orthogonal polarization states may be chosen to represent the polarization of the incident and scattered waves, it is advantageous to choose representations amenable to the coordinate system selected and to the method of specification of the incident and scattering directions. The electric field vector, the polarization of which will be described, lies in a plane perpendicular to the direction of propagation for both the incident and the scattered wave. Two convenient representations are in terms of the Cartesian or linear polarization states and in terms of the circular polarization states, both discussed in the preceeding section. First the polarization will be represented by the former method, and then

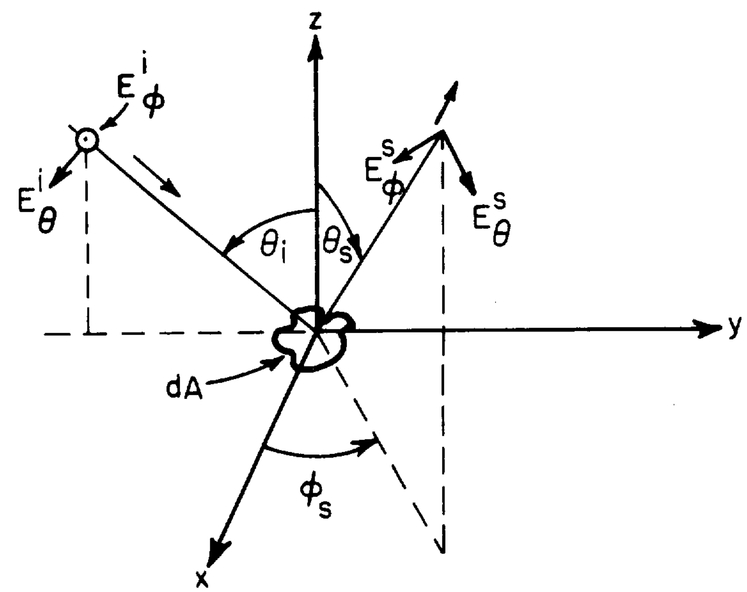

Fig. 2. Scattering of a polarized wave from a surface element dA.

the transformations of the preceding section will be used to convert to the circular representation.

The usual arrangement of the coordinate system is such that the vertical plane of incidence is the y-z plane, with the origin centered at some convenient point on dA. Then the directions of travel of the incident and scattered electromagnetic waves of interest (for example, those shown in Fig. 2) are completely specified by the angle of incidence, θ_i and the scattering angles, θ_s and ϕ_s.

The most obvious sets of Cartesian unit vectors to choose for these waves, both sets lying in planes perpendicular to the directions of propagation, are the unit vectors $\hat{\theta}_i$, $\hat{\phi}_i$ and $\hat{\theta}_s$, $\hat{\phi}_s$, i.e., the two sets of angular unit vectors in a spherical polar coordinate system along two radii extending out from the origin, one in the direction the incoming wave and the other in the direction of the outgoing wave. (Note: $\phi_i = 270°$ with the incident wave lying in the y-z plane). The components of the electric field of the incident and scattered waves along these unit vectors will be denoted by E^i_θ, E^i_ϕ and E^s_θ, E^s_ϕ, respectively.

Assume that the scattering process from the element dA is linear. Then, by superposition, one can assume first that the surface is illuminated only with a linearly polarized wave in the θ_i direction, E_θ^i. The field scattered in the direction (θ_s, ϕ_s) will have a component of polarization in the θ_s direction, i.e., E_θ^s. The relationship which exists between them for a given surface element, frequency, and orientation will then define a_{11}, where

$$E_\theta^s = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \; a_{11} E_\theta^i \; ;$$

The time dependent exponential, $e^{j\omega t}$, is dropped in this section. This same linearly polarized incident wave in the θ_i direction in general produces a component in the scattered wave polarized in the ϕ_s direction also, E_ϕ^s. The relationship between E_ϕ^s and the incident wave producing it, E_θ^i, defines a_{21}:

$$E_\phi^s = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \; a_{21} E_\theta^i \; .$$

The same process can be used to define a_{22} and a_{12}, where this time the incident wave is assumed polarized linearly only in the ϕ_i direction; this incident wave produces components polarized along each of the directions θ_s and ϕ_s in tne scattered wave.

$$E_\theta^s = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \; a_{12} E_\phi^i$$

and

$$E_\phi^s = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \; a_{22} E_\phi^i \; .$$

Since the scattering process is linear, superposition can be used to express the total scattered field due to an incident wave with both polarization components present. In matrix notation this relationship becomes

$$(3) \qquad \begin{bmatrix} E_\theta^s \\ E_\phi^s \end{bmatrix} = \frac{e^{-jkr_s}}{4\pi r_s^2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_\theta^i \\ E_\phi^i \end{bmatrix} \equiv \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} [A_L] \begin{bmatrix} E_\theta^i \\ E_\phi^i \end{bmatrix}$$

The a_{ij} are functions of the particular surface element, its orientation, the wavelength of the incident (and reflected) waves, and the incidence and scattering angles, θ_i, θ_s, and ϕ_s. The subscript 1 refers to θ and the subscript 2 refers to ϕ. The left-most subscript in a_{ij} refers to the scattered field component which is produced, while the right-most subscript refers to the incident field component under consideration.

There are four elements of this scattering matrix $[A_L]$. Each of these elements may be complex, in general. Therefore, there are altogether seven independent quantities associated with each scattering matrix: four magnitudes and three phase differences. (There are only three independent phase angles because a constant phase angle can be subtracted from each of the elements in the matrix and be absorbed by an appropriate shift in the time origin, making one of the elements of the matrix, a_{11}, for example, pure real.)

In the special case of back-scattering, where $\phi_s = 270°$ and $\theta_s = \theta_i$, it can be shown by the reciprocity theorem that $a_{12} = a_{21}$. Thus the number of independent quantities in the scattering matrix is reduced to five. At any angles of incidence and scattering other than the case of back scattering, i.e., bistatic scattering, no sucn relationship exists between a_{12} and a_{21}, however, for an arbitrary surface.

The reason for the peculiar definition of the a_{ij}, in which the factor $1/\sqrt{4\pi r_s^2}$ was expressed separately, will be evident now when the expressions for the scattering cross sections are examined. The scattering cross section can be defined for any combination of incident and scattered polarization states. Thus the scattering cross section relating incident power linearly polarized in the $\hat{\theta}_i$ direction to scattered power linearly polarized in the $\hat{\theta}_s$ direction from the surface element dA at angles θ_i, θ_s, ϕ_s is defined as follows:

$$\sigma_{\theta\theta}(\theta_i, \theta_s, \phi_s) = \sigma_{11}(\theta_i, \theta_s, \phi_s) = \frac{4\pi r_s^2 P_\theta^s}{P_\theta^i} = \frac{4\pi r_s^2 |E_\theta^s|^2}{|E_\theta^i|^2} = |a_{11}|^2 \quad .$$

The other scattering cross sections relating the ratio of power present in each of the polarization states of the scattered wave to those of the incident wave are similarly defined:

$$(4) \qquad \sigma_{\theta\theta}(\theta_i, \theta_s, \phi_s) = \sigma_{11} = |a_{11}|^2 ,$$

$$\sigma_{\theta\phi}(\theta_i, \theta_s, \phi_s) = \sigma_{12} = |a_{12}|^2 ,$$

$$\sigma_{\phi\theta}(\theta_i, \theta_s, \phi_s) = \sigma_{21} = |a_{21}|^2 ,$$

and

$$\sigma_{\phi\phi}(\theta_i, \theta_s, \phi_s) = \sigma_{22} = |a_{22}|^2 .$$

Again, the left subscript refers to the direction of polarization of the scattered wave, while the right subscript refers to the direction of polarization of the incident wave producing that component in the scattered wave.

One can define the scattering cross sections, which relate scattered power of any desired polarization state to the incident power producing it, of any other polarization state. For example, $\sigma_{RL}(\theta_i, \theta_s, \phi_s)$ relates scattered power, right circularly polarized, to incident power, left circularly polarized, which produces it. The formal definition is

$$\sigma_{RL}(\theta_i, \theta_s, \phi_s) = \frac{4\pi r_s^2 |E_R^s|^2}{|E_L^i|^2} = |a_{RL}|^2 .$$

A scattering matrix relating a scattered field, specified in the circular polarization states, to an incident field, also in the circular states, has not yet been discussed or defined. However, it appears that such a matrix would be of value, since, as seen above, the scattering cross sections for the circular states are defined simply as the magnitudes squared of the individual elements of such a matrix, in the same manner as they were for polarization specified in the Cartesian or linear polarization representation. Such a matrix one could certainly measure for a given surface element; however, since polarization specified in the circular representation is directly related to its specification in the Cartesian representation, one would suspect that the elements of the Cartesian or linear scattering matrix, $[A_L]$, would somehow be related to the elements of a circular scattering matrix, $[A_C]$. The exact relationship can be found easily by using the linear-to-circular transformations of the last section (Eq. (2)). The matrix desired has the following definition:

$$(5)\qquad \begin{bmatrix} E_R^s \\ \\ E_L^s \end{bmatrix} = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{RR} & a_{RL} \\ \\ a_{LR} & a_{LL} \end{bmatrix} \begin{bmatrix} E_R^i \\ \\ E_L^i \end{bmatrix} .$$

The scattering matrix previously discussed and defined is the following:

$$\begin{bmatrix} E_\theta^s \\ \\ E_\phi^s \end{bmatrix} = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_\theta^i \\ \\ E_\phi^i \end{bmatrix} ;$$

however, using relationship of Eq. (2), one has

$$\begin{bmatrix} E_\theta^s \\ \\ E_\phi^s \end{bmatrix} = [T_c]^{-1} \begin{bmatrix} E_R^s \\ \\ E_L^s \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_\theta^i \\ \\ E_\phi^i \end{bmatrix} = [T_c^*]^{-1} \begin{bmatrix} E_R^i \\ \\ E_L^i \end{bmatrix} . \quad +$$

Substituting these last two relationships into the previous one results in the following expression:

$$[T_c]^{-1} \begin{bmatrix} E_R^s \\ \\ E_L^s \end{bmatrix} = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} [T_c^*]^{-1} \begin{bmatrix} E_R^i \\ \\ E_L^i \end{bmatrix} .$$

Now, premultiply this equation by $[T_c]$ to give

\+ The conjugate matrix must be used for the incident field because of propagation radially inward toward the origin.

$$\begin{bmatrix} E_R^s \\ \\ E_L^s \end{bmatrix} = \frac{e^{-jkr_s}}{\sqrt{4\pi r_s^2}} \, [T_c] \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} [T_c^*]^{-1} \begin{bmatrix} E_R^i \\ \\ E_L^i \end{bmatrix} .$$

From the above, one can see that the elements of the circular scattering matrix of Eq. (5) are defined by

$$(6) \qquad \begin{bmatrix} a_{RR} & a_{RL} \\ \\ a_{LR} & a_{LL} \end{bmatrix} = [T_c] \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} [T_c^*]^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & -\frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ +\frac{j}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \end{bmatrix} .$$

The individual elements of the circular scattering matrix are

$$a_{RL} = \frac{a_{11} + a_{22} - j(a_{12} - a_{21})}{2} \;; \quad a_{RR} = \frac{a_{11} - a_{22} + j(a_{12} + a_{21})}{2}$$

$$a_{LL} = \frac{a_{11} - a_{22} - j(a_{12} + a_{21})}{2} \;; \quad a_{LR} = \frac{a_{11} + a_{22} + j(a_{12} - a_{21})}{2} .$$

From the first expression of Eq. (6) above, the inverse relationship, giving the linear scattering elements in terms of the circular scattering elements, can be found using matrix algebra: premultiply the equation by $[T_c]^{-1}$ and postmultiply it by $[T_c^*]$.

$$(7) \qquad \begin{bmatrix} a_{11} & a_{12} \\ \\ a_{21} & a_{22} \end{bmatrix} = [T_c]^{-1} \begin{bmatrix} a_{RR} & a_{RL} \\ \\ a_{LR} & a_{LL} \end{bmatrix} [T_c^*] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \\ \frac{-j}{\sqrt{2}} & \frac{j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} a_{RR} & a_{RL} \\ \\ a_{LR} & a_{LL} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-j}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} & \frac{+j}{\sqrt{2}} \end{bmatrix}$$

$$\therefore \quad a_{11} = \frac{a_{RL} + a_{RR} + a_{LL} + a_{LR}}{2} \;; \quad a_{12} = \frac{j(a_{RL} - a_{RR} + a_{LL} - a_{LR})}{2}$$

(7) cont. $$a_{21} = \frac{-j(a_{RL}+a_{RR}-a_{LL}-a_{LR})}{2}; \quad a_{22} = \frac{a_{RL}-a_{RR}-a_{LL}+a_{LR}}{2} .$$

From the relationships of Eq. (6) the circular scattering cross sections will be determined in terms of the linear scattering matrix elements.

$$\sigma_{RL} = |a_{RR}|^2 = \frac{1}{4}[a_{11} + a_{22} - j(a_{12} - a_{21})][a_{11} + a_{22} - j(a_{12} - a_{21})]^*$$

$$= \frac{1}{4}[a_{11} + a_{22} - j(a_{12} - a_{21})][a_{11}^* + a_{22}^* + j(a_{12}^* - a_{21}^*)];$$

$$\therefore \quad \sigma_{RL} = \frac{1}{4}[|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$

$$+ 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\} + 2\mathrm{Im}\{a_{11}^*a_{12} + a_{11}a_{21}^* + a_{22}^*a_{12} + a_{22}a_{21}^*\}] .$$

In like manner,

$$\sigma_{RR} = |a_{RL}|^2 = \frac{1}{4}[|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$

$$+ 2\mathrm{Re}\{a_{12}a_{21}^* - a_{11}a_{22}^*\} + 2\mathrm{Im}\{a_{12}^*a_{11} + a_{21}^*a_{11} + a_{12}a_{22}^* + a_{21}a_{22}^*\}],$$

$$\sigma_{LL} = |a_{RL}|^2 = \frac{1}{4}[|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$

$$+ 2\mathrm{Re}\{a_{12}a_{21}^* - a_{11}a_{22}^*\} + 2\mathrm{Im}\{a_{12}a_{11}^* + a_{21}a_{11}^* + a_{22}a_{12}^* + a_{22}a_{21}^*\}] ,$$

and

$$\sigma_{LR} = |a_{LL}|^2 = \frac{1}{4}[|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$

$$+ 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\} + 2\mathrm{Im}\{a_{12}^*a_{11} + a_{12}^*a_{22} + a_{21}a_{11}^* + a_{21}a_{22}^*\}] .$$

In the above equation, the expression Re{ } has the meaning "the real part of { }," and the expression Im{ } has the meaning "the imaginary part of { }". The above equations are rewritten below with the following substitutions: $|a_{11}|^2 = \sigma_{11}$, $|a_{22}|^2 = \sigma_{22}$, $|a_{12}|^2 = \sigma_{12}$, and $|a_{21}|^2 = \sigma_{21}$, and using the fact that $\mathrm{Im}\{x^*y\} = \mathrm{Im}\{-xy^*\}$.

(8)
$$\sigma_{RL} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + \sigma_{12} + \sigma_{21} + 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\}$$

$$+ 2\mathrm{Im}\{a_{11}a_{21}^* - a_{11}a_{12}^* + a_{22}a_{21}^* - a_{22}a_{12}^*\}] ,$$

$$\sigma_{RR} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + \sigma_{12} + \sigma_{21} - 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\}$$

$$+ 2\mathrm{Im}\{a_{11}a_{21}^* + a_{11}a_{12}^* - a_{22}a_{21}^* - a_{22}a_{12}^*\}] ,$$

$$\sigma_{LL} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + \sigma_{12} + \sigma_{21} - 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\}$$

$$-2\mathrm{Im}\{a_{11}a_{21}^* + a_{11}a_{12}^* - a_{22}a_{21}^* - a_{22}a_{12}^*\}] ,$$

and

$$\sigma_{LR} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + \sigma_{12} + \sigma_{21} + 2\mathrm{Re}\{a_{11}a_{22}^* - a_{12}a_{21}^*\}$$

$$- 2\mathrm{Im}\{a_{11}a_{21}^* - a_{11}a_{12}^* + a_{22}a_{21}^* - a_{22}a_{12}^*\}] .$$

The specification of the linear scattering cross sections in terms of the circular matrix elements may be done in a manner identical to that used above to derive Eq. (8), this time using relations of Eq. (7). The results are given below.

(9)
$$\sigma_{11} = \frac{1}{4}[\sigma_{RL} + \sigma_{LR} + \sigma_{RR} + \sigma_{LL} + 2\mathrm{Re}\{a_{RL}a_{LR}^* + a_{RR}a_{LL}^*\}$$

$$+ 2\mathrm{Re}\{a_{RL}a_{RR}^* + a_{RL}a_{LL}^* + a_{LR}a_{RR}^* + a_{LR}a_{LL}^*\}] ,$$

$$\sigma_{12} = \frac{1}{4}[\sigma_{RL} + \sigma_{LR} + \sigma_{RR} + \sigma_{LL} - 2\mathrm{Re}\{a_{RL}a_{LR}^* + a_{RR}a_{LL}^*\}$$

$$+ 2\mathrm{Re}\{-a_{RL}a_{RR}^* + a_{RL}a_{LL}^* + a_{LR}a_{RR}^* - a_{LR}a_{LL}^*\}] ,$$

(9) cont.

$$\sigma_{21} = \frac{1}{4}[\sigma_{RL} + \sigma_{LR} + \sigma_{RR} + \sigma_{LL} - 2\mathrm{Re}\{a_{RL}a_{LR}^{*} + a_{RR}a_{LL}^{*}\}$$

$$- 2\mathrm{Re}\{-a_{RL}a_{RR}^{*} + a_{RL}a_{LL}^{*} + a_{LR}a_{RR}^{*} - a_{LR}a_{LL}^{*}\}] ,$$

and

$$\sigma_{22} = \frac{1}{4}[\sigma_{RL} + \sigma_{LR} + \sigma_{RR} + \sigma_{LL} + 2\mathrm{Re}\{a_{RL}a_{LR}^{*} + a_{RR}a_{LL}^{*}\}$$

$$- 2\mathrm{Re}\{a_{RL}a_{RR}^{*} + a_{RL}a_{LL}^{*} + a_{LR}a_{RR}^{*} + a_{LR}a_{LL}^{*}\}] .$$

For the special case of back-scattering, where $a_{12} = a_{21}$ and $\sigma_{12} = \sigma_{21}$, Eq. (6) reveals that $a_{RL} = a_{LR}$. In this case, the relationships of Eq. (8) become

(10)

$$\sigma_{RL} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + 2\mathrm{Re}\{a_{11}a_{22}^{*}\}] ,$$

$$\sigma_{RR} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + 4\sigma_{12} - 2\mathrm{Re}\{a_{11}a_{22}^{*}\} + 4\mathrm{Im}\{a_{11}a_{12}^{*} - a_{22}a_{12}^{*}\}] ,$$

$$\sigma_{LL} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + 4\sigma_{12} - 2\mathrm{Re}\{a_{11}a_{22}^{*}\} - 4\mathrm{Im}\{a_{11}a_{12}^{*} - a_{22}a_{12}^{*}\}] ,$$

and

$$\sigma_{LR} = \frac{1}{4}[\sigma_{11} + \sigma_{22} + 2\mathrm{Re}\{a_{11}a_{22}^{*}\}] .$$

The main point of interest evident from the above equations is that $\sigma_{RL} = \sigma_{LR}$ for the case of back-scattering, just as $\sigma_{12} = \sigma_{21}$ in this case. The expressions of Eq. (9) may be simplified in this case also to the following:

(11)

$$\sigma_{11} = \frac{1}{4}[4\sigma_{RL} + \sigma_{RR} + \sigma_{LL} + 2\mathrm{Re}\{a_{RR}a_{LL}^{*}\}$$

$$+ 4\mathrm{Re}\{a_{RL}a_{RR}^{*} + a_{RL}a_{LL}^{*}\}],$$

$$\sigma_{12} = \frac{1}{4}[\sigma_{RR} + \sigma_{LL} - 2\mathrm{Re}\{a_{RR}a_{LL}^{*}\}] ,$$

(11) cont. $$\sigma_{21} = \frac{1}{4}[\sigma_{RR} + \sigma_{LL} - 2\mathrm{Re}\{a_{RR}a_{LL}^*\}] ,$$

and

$$\sigma_{22} = \frac{1}{4}[4\sigma_{RL} + \sigma_{RR} + \sigma_{LL} + 2\mathrm{Re}\{a_{RR}a_{LL}^*\} - 4\mathrm{Re}\{a_{RL}a_{RR}^* + a_{RL}a_{LL}^*\}] .$$

The significant point to note about all of the above transformations is that the relationship between the linear scattering matrix, $[A_L]$, and the circular scattering matrix, $[A_C]$, is a linear transformation consisting of a matrix premultiplication and matrix postmultiplication, as seen in Eqs. (6) and (7). This means that the elements of $[A_L]$ can be expressed solely in terms of the elements of $[A_C]$, and conversely. No such relationship exists between the circular scattering cross sections and the linear ones, however. This means that the circular scattering cross sections can never be expressed or found solely from a knowledge of the linear scattering cross sections, and conversely. The reason for this is apparent; the scattering matrices, $[A_L]$ and $[A_C]$, each contain seven independent pieces of information about the polarization - transforming properties of the surface. These seven pieces of information are all needed to completely characterize the polarization - transforming properties of the surface. However, the four scattering cross sections for each case contain only four pieces of information; they carry no information about the three independent phase differences between the elements of the scattering matrices. Therefore they do not completely characterize the polarization-transforming properties of a surface in themselves, and one needs more information about the phase differences of the matrix elements in order to relate the circular scattering cross sections to the linear ones, and conversely.

The important relationships discussed in this section are the definitions of the linear and circular scattering matrices of a surface, given in Eqs. (3) and (5). The relationship between the elements of the scattering matrix and the scattering cross sections of a surface are given in Eq. (4). The transformations between the elements of the circular scattering matrix and the linear scattering matrix are given in Eqs. (6) and (7). The relationships between the circular scattering cross sections and the linear cross sections are given in Eq. (8), while the converse relationships are given in Eq. (9). Then these scattering cross section relationships are specialized for the case of back-scattering, and these results are seen in Eqs. (10) and (11). In the case of back-scattering, it was seen that $a_{12} = a_{21}$, $\sigma_{12} = \sigma_{21}$, and $a_{RL} = a_{LR}$, $\sigma_{RL} = \sigma_{LR}$. (See refs. 1, 2, and 6.)

III. ANTENNA VECTOR HEIGHT METHOD FOR SPECIFICATION OF RECEPTION AND TRANSMISSION OF POLARIZED WAVES

There are several methods of specifying the transmitted electric field from an antenna and representing its polarization, but the concept of vector height is one of the most useful[5, 8, 10]. Its use is restricted primarily to the far-zone region only, where the radial component of the electric field becomes negligible. In this case the components of the electric field can be resolved into transverse components along the $\hat{\theta}$ and $\hat{\phi}$ directions, where θ and ϕ are the polar and azimuthal angles in spherical polar coordinates, and $\hat{\theta}$ and $\hat{\phi}$ are the unit vectors in the directions of increasing θ and ϕ and hence perpendicular to the radial unit vector and direction of propagation. The antenna is assumed to be located at the origin. The relationship giving the far-zone electric field in terms of the vector height for any antenna located at the origin is the following (see Fig. 3):

$$\overline{E} = j\,\frac{Z_o I\overline{h}}{2\lambda\, r}\, e^{-jkr}, \tag{12}$$

where

Z_o = intrinsic impedance of free space for plane waves ($120\pi\Omega$),

r = radial distance from antenna at the origin,

I = terminal current at the antenna,

λ = wavelength, and

$\overline{h} = \overline{h}(\theta, \phi) \equiv$ vector height of the antenna.

The above definition illustrates that the only parameter which depends upon the particular antenna chosen is $\overline{h}$, the vector height. The quantities I and λ are considered input variables to the antenna capable of being controlled by the user. Vector height, $\overline{h}$, may be a function of λ also.

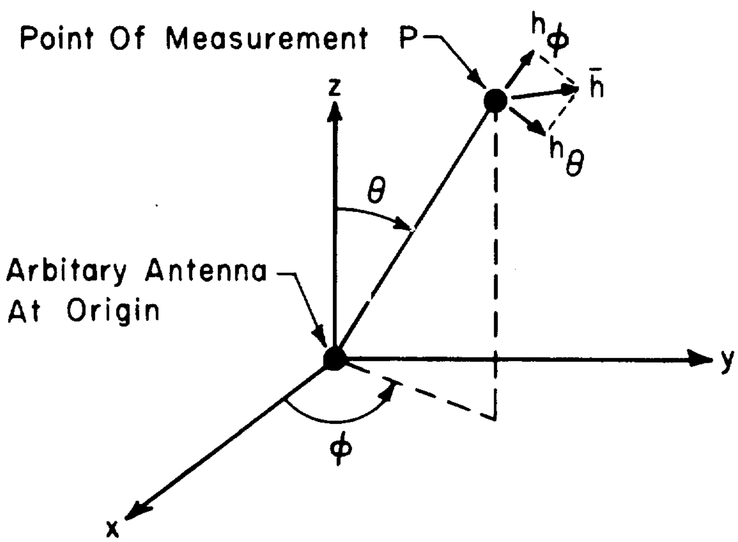

Fig. 3. Vector height of antenna at point P.

The vector height, $\bar{h}$, can be plotted as a function of θ and ϕ in much the same manner as the gain function of an antenna, except that in this case the components of $\bar{h}$ specify an amplitude and phase, and not merely an intensity. There are two components of $\bar{h}$ (i.e., h_θ and h_ϕ) and each is, in general, a function of θ and ϕ; and each is, in general, complex, having an amplitude and a phase angle. While the gain function for an antenna is easily measured, since intensity is an easily measurable quantity, the vector height of an antenna is not easily measurable from any practical experiment, since measurement of the vector height requires observing and preserving the phase difference between the components of $\bar{h}$. Therefore in most cases it is easiest to calculate the vector height theoretically for a particular antenna rather than attempt to measure it, even though the results may be highly idealized.

As examples, the theoretical formulas for the vector height of three particular simple antennas are stated here[13]:

(a) for a short linear element of length 2L centered at the origin and oriented along the polar, or z-axis,

$$\bar{h} = L \sin\theta \cdot \hat{\theta} + 0 \cdot \hat{\phi} ;$$

(b) for a half-wave dipole antenna parallel to the z-axis and centered at the origin,

$$\bar{h} = \frac{\lambda}{\pi} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\,\hat{\theta} + 0 \cdot \hat{\phi}\ ;\ \text{and}$$

(c) for a small loop of area S with its axis along the z-axis and centered at the origin,

$$\bar{h} = 0 \cdot \hat{\theta} - j\,\frac{2\pi S}{\lambda}\,\sin\theta \cdot \hat{\phi}\,.$$

(Note: throughout this section it is assumed all voltages, currents, and fields are sinusoidal with respect to time, and therefore the factor $e^{j\omega t}$ has been omitted from equations.)

Instead of representing the polarized electric field and vector height in vector form it is more convenient in many instances to represent them in matrix form as follows:

$$\bar{E} = [E] = \begin{bmatrix} E_\theta \\ E_\phi \end{bmatrix} = \frac{jZ_o I e^{-jkr}}{2\lambda r}\,[h] = \frac{jZ_o I e^{-jkr}}{2\lambda r} \begin{bmatrix} h_\theta \\ h_\phi \end{bmatrix}\,.$$

Although h_θ and h_ϕ may be complex in general, it is always possible to make one of them, say h_θ, pure real by subtracting the proper constant angle from each component and absorbing it in a shift in the time origin. Thus,

$$(14) \qquad h_\theta = |h_\theta|\ ;\quad h_\phi = |h_\phi|\,e^{j\delta_1}\,.$$

In order to extend the application of the vector height concept to the reception of elliptically polarized waves by an antenna of a particular $\bar{h}$, consider a plane wave incident from angles θ_i and ϕ_i upon such an antenna located at the coordinate origin. The wave has only transverse components and therefore may be expressed as

$$\overline{E}^i = e^{ikr_1} \begin{bmatrix} E^i_\theta \\ E^i_\phi \end{bmatrix} .$$

Here, r_1 is the radial distance measured along the direction of propagation. As before, E^i_θ can be made pure real, i.e.,

$$E^i_\theta = |E^i_\theta| \; ; \qquad E^i_\phi = |E^i_\phi| \, e^{-j\delta_2} \quad . \tag{15}$$

The minus sign was used here in contrast to the plus sign used in the definition of phase difference in Eq. (14). The reason for this convention is so that when δ_1 and δ_2 lie in the same quadrant, Eqs. (14) and (15) represent the same direction of rotation of the electric field vector to an observer looking in the direction from which the waves are coming. The difference in sign therefore results because in Eq. (14) for the vector height of a transmitted wave, the direction of propagation is taken to be away from the antenna; in Eq. (15) the wave is propagating toward the antenna.

In obtaining the open-circuit terminal voltage produced by the incoming wave upon the antenna located at the coordinate origin, the θ and ϕ components of the incident wave are considered separately and the resulting voltages are then added to give the total voltage. This is possible because the reception and transmission by the antenna is assumed to be a linear process, and therefore superposition may be applied.

By the Reciprocity Theorem[14], the open-circuit voltage produced across the terminals of the antenna by the θ component of the incident wave acting alone is given by

$$V' = h_\theta E^i_\theta = |h_\theta| \, |E^i_\theta| \quad .$$

Similarly, the voltage at the terminals resulting from the ϕ component of the incident wave acting alone is given by

$$V'' = h_\phi \, E^i_\phi = |h_\phi| \, |E^i_\phi| \, e^{j(\delta_1 - \delta_2)} \, .$$

Therefore, by superposition, the total open-circuit terminal voltage is given by

$$(16) \qquad V = V' + V'' = h_\theta E_\theta^i + h_\phi E_\phi^i = \begin{bmatrix} h_\theta h_\phi \end{bmatrix} \begin{bmatrix} E_\theta^i \\ E_\phi^i \end{bmatrix} = [h]^T [E^i] = \bar{h} \cdot \bar{E}^i .$$

The above equation illustrates the different ways of representing the reception, i.e., by a sum of algebra products, by matrix product, and by a vector dot product; all say exactly the same thing. The superscript "T" above a matrix refers to the transpose of the matrix, e.g., the transpose of the column matrix of this case is a row matrix. As seen previously and as will be more evident later, the matrix method proves to be the most convenient, especially when considering the reception of scattered waves.

In this section, the concept of vector height of an antenna was discussed, and the polarized electric field transmitted by an antenna in terms of its vector height and terminal current was given in Eq. (12). Several examples of vector heights for particular simple antennas were cited in the relations of Eq. (13). The extension of the vector height of an antenna to its reception of a general elliptically polarized wave was discussed and the terminal voltage of the receiving antenna is given in Eq. (16).

IV. TRANSMISSION, SCATTERING, AND RECEPTION OF ELLIPTICALLY POLARIZED WAVES

In the previous sections, the representation, transmission, scattering, and reception of elliptically polarized radiation wave were all considered separately. In this section, these concepts will be combined; an expression for the voltage (and power) induced in the receiving antenna after transmission by a separate antenna and after scattering by an arbitrary surface will be found, as a function of the current at the terminals of the transmitting antenna. This voltage will be a function of the polarization properties (vector height) of the antennas and their particular orientation about the lines of propagation to the surface, r_t and r_r. It will also be a function on the scattering properties of the surface (the scattering matrix elements) and the orientation of this element along with the directions of incidence and scattering from the surface to the transmitting and receiving antenna (see Fig. 4). Finally, it is a function of the frequency of the radiation.

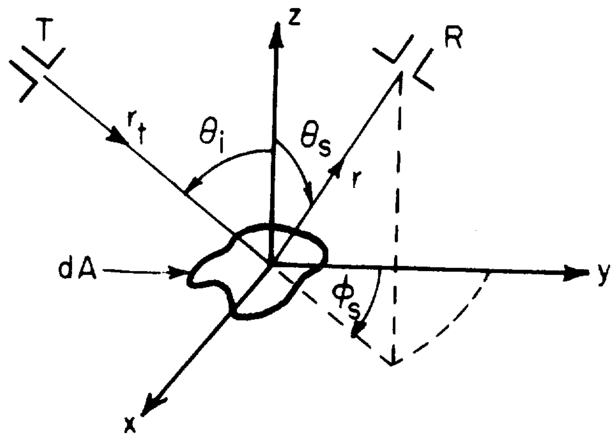

Fig. 4. Bistatic scattering

It is assumed that the medium is homogeneous and isotropic, and that the distances from antennas to the surface are large enough so that the far-zone approximations are valid. For the present, it is assumed that the surface element and antennas are stationary with respect to one another.

The motivation for such an analysis seems quite evident. Given all the properties of the antennas, their orientation, and the properties of the scattering surface for a given frequency, find the voltage induced in the receiving antenna for a given current at the transmitting antenna terminals. However, the main purpose of these studies is somewhat the converse of the above statement; i.e., given the properties of the antennas and their orientation, what properties of the surface (involving the scattering matrix elements) can be determined from measurement of the receiving antenna voltage (or power) for a given transmitting antenna current and frequency. For this purpose, it is convenient to choose the simplest possible antennas (having easily describable vector heights) in the measurements, so that mathematical determination is not too burdensome. Therefore, linear-favoring antennas will be considered first; such an antenna which transmits a linearly polarized wave is the half-wave dipole antenna, discussed in Section III; its vector height is given in the second expression of Eq. (13). Then, circular-favoring antennas will be employed also because linear antennas alone cannot describe completely the surface properties.

Consider now that the transmitting antenna is a half-wave dipole aligned with its axis pointing toward the surface (at the coordinate origin) along r_t, and so that it lies in the plane of incidence, the y-z plane. Then let the receiving antenna be aligned with its axis pointing toward the surface along r_r, and so that it is rotated about its axis by an angle α from the plane of scattering (α is positive for rotation in a clockwise direction when looking toward the antenna from the surface element).

The transmitting antenna vector height, expressed in the coordinate system shown above is,

$$\bar{h}_t = \begin{bmatrix} \frac{\lambda}{\pi} \\ 0 \end{bmatrix} ,$$

where the upper element refers to the component in the $+\hat{\theta}_i$ direction. The vector height for the receiving antenna is

$$\bar{h}_r = \begin{bmatrix} \frac{\lambda}{\pi} \cos \alpha \\ \frac{-\lambda}{\pi} \sin \alpha \end{bmatrix} ,$$

where the upper element refers to the component along the $+\hat{\theta}_s$ direction and the lower element refers to the $+\hat{\phi}_s$ component.

Therefore, using the scattering matrix concept, one can immediately write down the value of the open-circuit terminal voltage at the receiving antenna:

$$(17) \qquad V_v = j \frac{Z_o I}{2\lambda r_t} [h_r]^T [A] [h_t]$$

$$= j \frac{Z_o I}{2\lambda r_t} \cdot \frac{\lambda}{\pi} \cdot \frac{\lambda}{\pi} \frac{1}{\sqrt{4\pi r_r^2}} \begin{bmatrix} \cos \alpha & -\sin \alpha \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} ;$$

$$V_v = j \frac{Z_o I \lambda}{4 r_t r_s \pi^{5/2}} (a_{11} \cos \alpha - a_{21} \sin \alpha) .$$

Now, assume that the transmitting antenna is rotated about its axis by an angle $\pi/2$ so that its vector height has the following form:

$$\bar{h}_t = \begin{bmatrix} 0 \\ \frac{\lambda}{\pi} \end{bmatrix} .$$

In this case the voltage at the receiving antenna is given by

$$(18) \qquad V_h = \frac{jZ_o I\lambda}{4r_r r_s \pi^{5/2}} \; (a_{12}\cos\alpha - a_{22}\sin\alpha).$$

In these two expressions, Eqs. (17) and (18), are found all the four elements of the scattering matrix. It would appear at first glance that from an exact knowledge of V_v, V_h, and I for a set of measurements at several values of α, one could exactly determine a_{11}, a_{12}, a_{21}, and a_{22}. However, in practice this involves an exact knowledge of the magnitude of V_v, V_h, and I, and also an exact measure of the phase differences between I and V_v, V_h. It is the latter information which is difficult to obtain accurately in practice. Therefore it is generally more convenient to measure magnitude information, or power. Such "power", or intensity of the voltage, is defined as follows:

$$P = |V|^2 = V \cdot V^* .$$

Thus the absolute value squared of equations (17) and (18) gives an expression for the power received in terms of transmitted power and angle of rotation of the receiving antenna, α, at both transmitting antenna positions.

$$(19) \qquad P_v = V_v \cdot V_v^* = \left(\frac{Z_o\lambda}{4r_t r_s \pi^{5/2}}\right)^2 |I|^2$$

$$(a_{11}\cos\alpha - a_{21}\sin\alpha)(a_{11}^*\cos\alpha - a_{21}^*\sin\alpha),$$

$$P_v = \left(\frac{Z_o\lambda}{4r_t r_s \pi^{5/2}}\right)^2 |I|^2 (|a_{11}|^2\cos^2\alpha - 2\mathrm{Re}\{a_{11}a_{21}^*\}$$

$$\cos\alpha\sin\alpha + |a_{21}|^2 \sin^2\alpha),$$

or

$$P_v = \left(\frac{Z_o \lambda}{4 r_t r_s \pi^{5/2}}\right)^2 |I|^2 (|a_{11}|^2 \cos^2 \alpha$$

$$- 2|a_{11}||a_{21}| \cos \delta_1 \cos \alpha \sin \alpha + |a_{21}|^2 \sin^2 \alpha),$$

where δ_1 = phase difference between a_{11} and a_{21}.

(20) $$P_h = V_h \cdot V_h^* = \left(\frac{Z_o \lambda}{4 r_t r_s \pi^{5/2}}\right)^2 |I|^2$$

$$(a_{12} \cos \alpha - a_{22} \sin \alpha)(a_{12}^* \cos \alpha - a_{22}^* \sin \alpha),$$

$$P_h = \left(\frac{Z_o \lambda}{4 r_t r_s \pi^{5/2}}\right)^2 |I|^2 (|a_{12}|^2 \cos^2 \alpha$$

$$- 2\mathrm{Re}\{a_{12} a_{22}^*\} \cos \alpha \sin \alpha + |a_{22}|^2 \sin^2 \alpha),$$

or

$$P_h = \left(\frac{Z_o \lambda}{4 r_t r_s \pi^{5/2}}\right)^2 |I|^2 (|a_{12}|^2 \cos^2 \alpha$$

$$-2|a_{12}||a_{22}| \cos \delta_2 \cos \alpha \sin \alpha + |a_{22}|^2 \sin^2 \alpha) ,$$

where δ_2 = phase difference between a_{12} and a_{22}.

It is apparent from Eq. (19) that after three measurements of power at properly chosen receiving antenna angle (e.g., $\alpha = 0$, $\pi/2$, and $\pi/4$), the quantities $|a_{11}|$, $|a_{21}|$, and $\cos \delta_1$ can be determined. After three similar measurements with the transmitting antenna horizontal and by use of Eq. (20), D-4, quantities $|a_{12}|$, $|a_{22}|$, and $\cos \delta_2$ can be determined.

In addition, it is interesting to note the form of Eqs. (19) and (20); if one considers the quantities $|a_{11}|$, $|a_{21}|$, and $\cos \delta_1$ as parameters of a given surface and plots P_n as a function of α, rotation of the receiving antenna, the polar plot is a variation of a familiar figure, the ellipse. The general equation for an ellipse centered at the origin and with major axes, in general, inclined is

$$\frac{x^2}{A^2 \sin^2 \delta} + \frac{y^2}{B^2 \sin^2 \delta} - \frac{2xy \cos \delta}{AB \sin^2 \delta} = 1.$$

Upon conversion to polar coordinates this becomes

$$\frac{r^2 \cos^2 \alpha}{A^2 \sin^2 \delta} + \frac{r^2 \sin^2 \alpha}{B^2 \sin^2 \delta} - \frac{2r^2 \cos \alpha \sin \alpha \cos \delta}{AB \sin^2 \delta} = 1 ,$$

or

$$\frac{1}{r^2} = \left(\frac{1}{A^2 \sin^2 \delta}\right) \cos^2 \alpha - 2\left(\frac{1}{AB\sin^2 \delta} \cos \delta\right) \cos \alpha \sin\alpha$$

$$+ \left(\frac{1}{B^2 \sin^2 \delta}\right) \sin^2 \alpha .$$

Now if one substitutes

$$|a_{11}| = \frac{1}{A \sin \delta} ,$$

$$|a_{21}| = \frac{1}{B \sin \delta} ,$$

and

$$\frac{1}{r^2} = P_V = |V_V|^2 ,$$

one arrives at Eq. (19). Therefore, if one were to take the inverse of the square root of P_V as a function of α, he would obtain an ellipse. Several of these figures are shown for various values of

$$\frac{|a_{11}|}{|a_{21}|}$$

and $\cos \delta_1$ in Fig. 5.

The following quantities in the scattering matrix for the surface have been determined from power measurements thus far, viz., $|a_{11}|$,

$|a_{21}|$, $|a_{22}|$, $|a_{12}|$, $\cos \delta_1$ and $\cos \delta_2$. The remaining problem is to determine the signs of angles δ_1 and δ_2 and the remaining independent phase difference in the scattering matrix. The quantities $\cos \delta_1$ and $\cos \delta_2$ do not uniquely determine δ_1 and δ_2. The remaining problem at first seems trivial: make a few more measurements at different transmitting antenna angles than 0 and $\pi/2$. However, this problem, upon investigation, is more complex than it seems. Such measurements yield involved expressions containing the remaining phase angle in question, but there is still an ambiguity in the sign of various phase angles. There is some question as to whether all of these independent phase differences and their proper phase angles can be determined by transmitting and receiving strictly linear polarization, as previously discussed. If such a determination is possible, it is, at best, an involved process.

However, if one employs circularly polarized antennas, these ambiguities can be resolved quickly. There are many possible schemes involving circularly polarizing antennas which can do the job, and the schemes chosen here are by no means the only possible one. However, they are simple, both physically and in the interpretation of the results. Assume that from the previously described linear antennas, $\cos \delta_1$ and $\cos \delta_2$ have been determined along with the four magnitudes of the scattering matrix elements. Left to be determined first are $\sin \delta_1$ and $\sin \delta_2$, which then uniquely determine δ_1 and δ_2. Finally, δ_3 must be uniquely determined. In the next paragraph, a simple scheme for finding $\sin \delta_1$ and $\sin \delta_2$ will be described.

In this system, a linear polarizing transmitting antenna, along with a circular polarizing receiving antenna, will be used in measurement of power scattered from a surface. For generality, assume that the transmitting antenna is rotatable by an angle α from $\hat{\theta}_i$ in a clockwise direction when looking toward the surface along the direction of propagation of the incident wave; both antenna vector heights are normalized and given below.

$$\overline{h}_t = \begin{bmatrix} \cos \alpha \\ \sin \alpha \end{bmatrix} \quad ; \qquad \overline{h}_r = \begin{bmatrix} 1 \\ \pm j \end{bmatrix} .$$

The top sign on the lower element of the receiving antenna indicates that the wave emitted by this antenna, if transmitting, is left circular; the bottom sign indicates right circular. The voltage received, therefore, is given by the following:

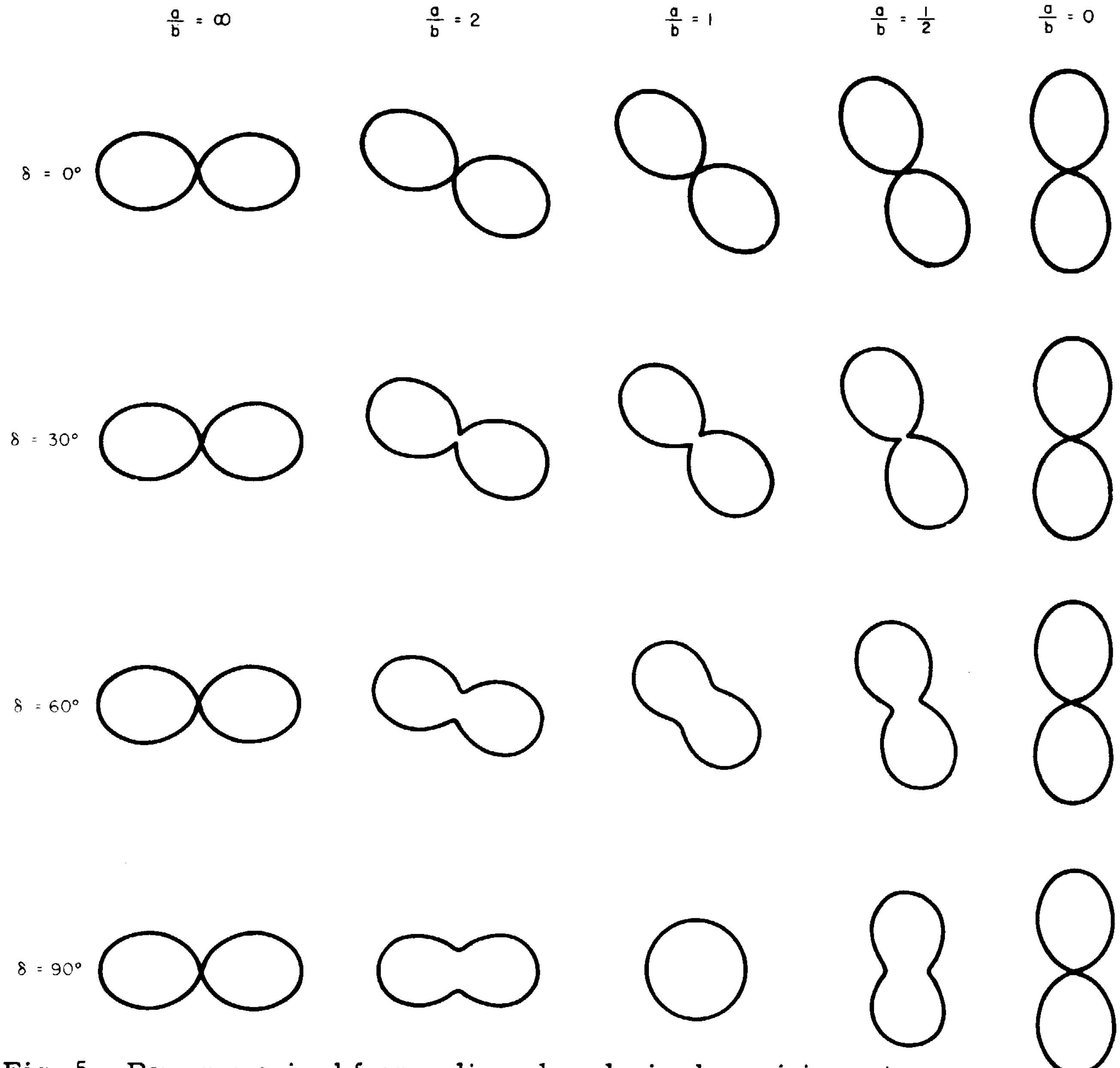

Fig. 5. Power received from a linearly polarized receiving antenna versus antenna angle, θ, for various values of scattering surface parameters, a, b, and δ.

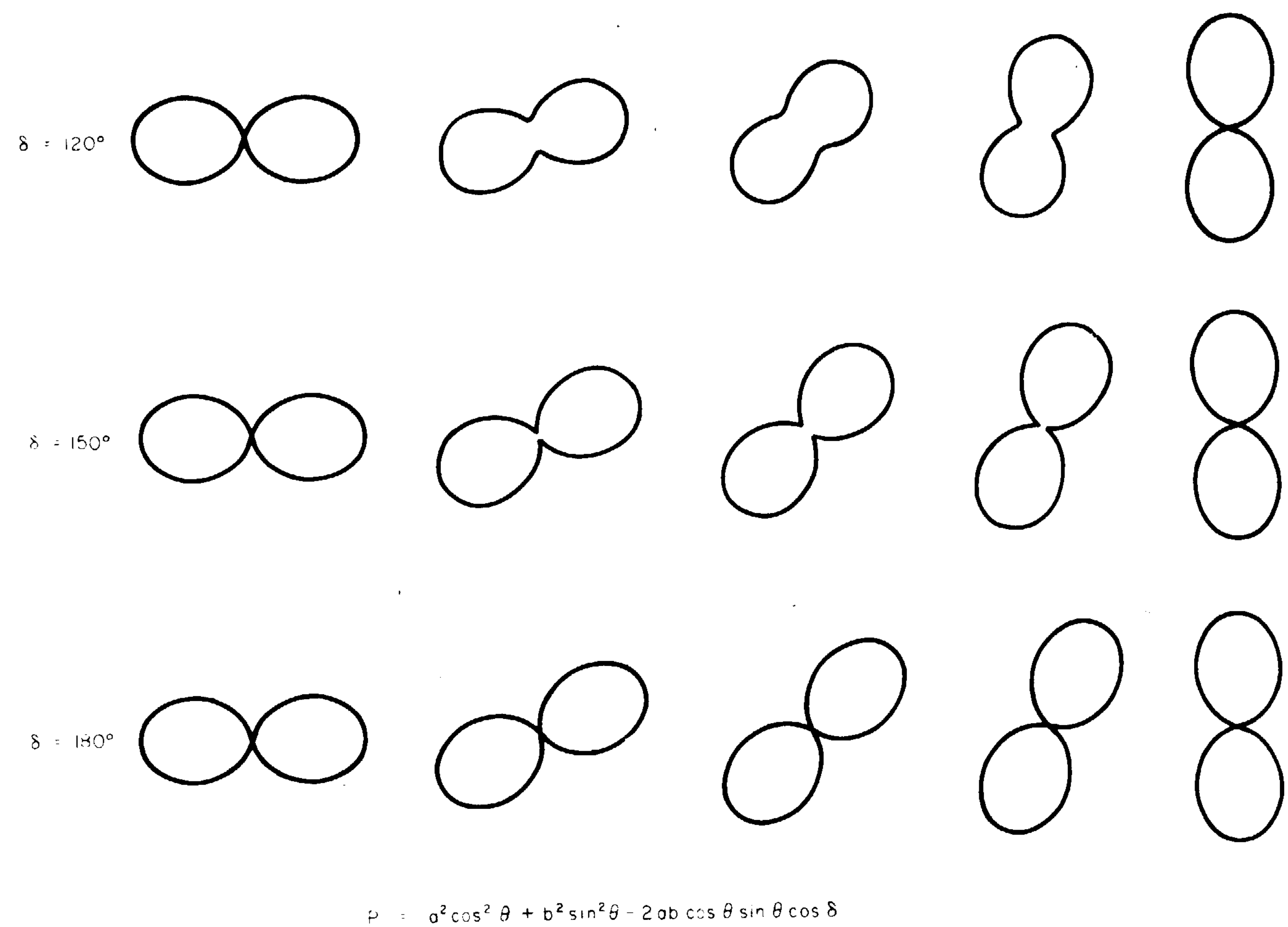

Fig. 5. Power received from a linearly polarized receiving antenna versus antenna angle, θ, for various values of scattering surface parameters, a, b, and δ (cont).

$$V_{LC} = k[h_r]^T[a][h_t] = k[a_{11}\cos\alpha + a_{12}\sin\alpha \pm ja_{21}\cos\alpha \pm ja_{22}\sin\alpha].$$

Therefore the power received is proportional to the voltage squared and is given below.

$$\begin{aligned} P_{LC} = V_{LC} \cdot V_{LC}^* = K\{ & a_{11}a_{11}^*\cos^2\alpha + a_{12}a_{12}^*\sin^2\alpha \\ & + a_{11}a_{12}^*\cos\alpha\sin\alpha + a_{12}a_{11}^*\cos\alpha\sin\alpha \\ & + a_{21}a_{21}^*\cos^2\alpha + a_{22}a_{22}^*\sin^2\alpha \\ & + a_{21}a_{22}^*\cos\alpha\sin\alpha + a_{22}a_{21}^* + \cos\alpha\sin\alpha \\ & \mp ja_{11}a_{21}^*\cos^2\alpha \mp ja_{12}a_{21}^*\cos\alpha\sin\alpha \\ & \mp ja_{11}a_{22}^*\cos\alpha\sin\alpha \mp ja_{12}a_{22}^*\sin^2\alpha \\ & \pm ja_{21}a_{11}^*\cos^2\alpha \pm ja_{21}a_{12}^*\cos\alpha\sin\alpha \\ & \pm a_{11}a_{22}^*\cos\alpha\sin\alpha \pm ja_{12}a_{22}\sin^2\alpha\}. \end{aligned}$$

The angles as defined in Eq. (24) are used to simplify the above expression.

$$\begin{aligned} \text{(21)} \qquad P_{LC} = K\{ & [|a_{11}|^2 + |a_{21}|^2]\cos^2\alpha + [|a_{12}|^2 + |a_{22}|^2]\sin^2\alpha \\ & + 2[|a_{11}||a_{12}|\cos(\delta_3 - \delta_1) + |a_{21}||a_{22}|\cos(\delta_3 - \delta_2)]\cos\alpha\sin\alpha \\ & \mp 2|a_{11}||a_{21}|\sin\delta_1\cos^2\alpha \mp 2|a_{12}||a_{22}|\sin\delta_2\sin^2\alpha \\ & \mp 2[|a_{12}||a_{21}|\sin\delta_3 + 2|a_{11}||a_{22}|\sin(\delta_1 + \delta_2 - \delta_3)]\cos\alpha\sin\alpha\}. \end{aligned}$$

Although the above expression for received power is not simple mathematically as it stands, it can be simplified by choosing α judiciously. If $\alpha = 0$ (transmitting antenna vertical), then

$$\text{(22)} \qquad P_{LC} = K\{|a_{11}|^2 + |a_{21}|^2 \mp 2|a_{11}||a_{21}|\sin\delta_1\};$$

if $\alpha = \pi/2$ (transmitting antenna horizontal), then

$$P_{LC} = K\{ |a_{12}|^2 + |a_{22}|^2 \mp 2|a_{12}||a_{22}| \sin \delta_2\} \ . \tag{23}$$

The above two expressions are quite simple and readily yield $\sin \delta_1$ and $\sin \delta_2$ after the four magnitudes are known. The upper sign in the last term refers to a receiving antenna which itself transmits right circular (and therefore best receives left circular) polarization.

An alternative scheme can be used when the transmitting antenna is circular and the receiving antenna is linear. The angle α here denotes clockwise rotation of the receiving antenna from $\hat{\theta}_s$ (vertical) when looking toward the antenna from the surface in the direction of propagation; the upper sign on the lower element of the transmitting antenna vector height again represents an antenna which transmits a left circular wave. The results are merely stated below.

$$\bar{h}_t = \begin{bmatrix} 1 \\ \pm j \end{bmatrix} \quad ; \quad \bar{h}_r = \begin{bmatrix} \cos \alpha \\ -\sin \alpha \end{bmatrix} ,$$

$$\begin{aligned} \therefore \ P_{CL} = K\{ & [|a_{11}|^2 + |a_{21}|^2]\cos^2 \alpha + [|a_{12}|^2 + |a_{22}|^2]\sin^2 \alpha \\ & - 2[|a_{11}||a_{21}|\cos \delta_1 + |a_{22}||a_{12}|\cos \delta_2]\cos \alpha \sin \alpha \\ & \pm 2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1)\cos^2 \alpha \pm 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)\sin^2 \alpha \\ & \mp [2|a_{11}||a_{22}|\sin(\delta_3 - \delta_2 - \delta_1) + 2|a_{12}||a_{21}|\sin \delta_3]\cos \alpha \sin \alpha\} \ . \end{aligned}$$

For $\alpha = 0$

$$P_{CL} = K\{|a_{11}|^2 + |a_{21}|^2 \pm 2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1)\} \ ;$$

for $\alpha = \pi/2$

$$P_{CL} = K\{ |a_{12}|^2 + |a_{22}|^2 \pm 2|a_{12}||a_{22}|\sin(\delta_3 - \delta_2)\} \ .$$

It is not immediately obvious whether the above two expressions can uniquely determine δ_3 if all the other quantities are known; it might turn out that each of these two expressions yields the same two choices for δ_3, which does not eliminate the ambiguity.

There is one remaining scheme to be investigated which will definitely clear up the ambiguity in sign of δ_3 (and also can be used as an alternative to the above method for resolving the ambiguity in δ_1 and δ_2). This involves the use of circular polarizing antennas for both transmitting and receiving. In this case it is easier to describe polarization states and the scattering matrix in terms of the circular representation. Thus, for antennas, both of which transmit right circular, we have

$$\overline{h}_t = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; ; \overline{h}_r = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad ; \quad [a] = \begin{bmatrix} a_{RR} & a_{RL} \\ a_{LR} & a_{LL} \end{bmatrix} \quad .$$

Then the voltage at the receiving antenna can be written as

$$V_{C_{RR}} = k I \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_{RR} & a_{RL} \\ a_{LR} & a_{LL} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = k I \, a_{RR}$$

and

$$P_{C_{RR}} = V_{C_{RR}} \cdot V_{C_{RR}}^{*} = k^2 \, |I|^2 \, |a_{RR}|^2 = k^2 \, |I|^2 \, \sigma_{RR} \, .$$

In like manner, a left circular transmitting antenna and right circular receiving antenna yield a power return

$$P_{C_{RL}} = k^2 \, |I|^2 \, \sigma_{RL} \quad .$$

The other combinations of circular antennas give similar results. The circular scattering cross sections can then be expressed in terms of the linear scattering matrix elements, as given in Eq. (8). At this point it is convenient to formally define and express the angles between all of the elements of the scattering matrix.

$$(24)\quad \angle a_{21} - \angle a_{11} \equiv \delta_1 \qquad ; \mathrm{Re}\{a_{12}a_{21}^*\} = |a_{12}||a_{21}|\cos\delta_3 ,$$

$$\angle a_{22} - \angle a_{12} \equiv \delta_2 \qquad ; \mathrm{Re}\{a_{11}a_{22}^*\} = |a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1),$$

and

$$\angle a_{21} - \angle a_{12} \equiv \delta_3 \qquad ; \mathrm{Im}\{a_{11}a_{21}^*\} = -|a_{11}||a_{21}|\sin\delta_1 ;$$

$$\therefore \angle a_{12} - \angle a_{11} = \delta_1 - \delta_3 \qquad ; \mathrm{Im}\{a_{11}a_{12}^*\} = +|a_{11}||a_{12}|\sin(\delta_3 - \delta_1),$$

$$\therefore \angle a_{21} - \angle a_{22} = \delta_3 - \delta_2 \qquad ; \mathrm{Im}\{a_{22}a_{21}^*\} = -|a_{22}||a_{21}|\sin(\delta_3 - \delta_2),$$

and

$$\angle a_{22} - \angle a_{11} = \delta_1 + \delta_2 - \delta_3 \ ; \mathrm{Im}\{a_{22}a_{12}^*\} = |a_{22}||a_{12}|\sin\delta_2 .$$

Using the above relations the power received by the various combinations of the circular antennas can be written as follows:

$$(25)\qquad P_{C_{RL}} = \left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$+ 2|a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1) - 2|a_{12}||a_{21}|\cos\delta_3$$
$$-2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) - 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)$$
$$-2|a_{11}||a_{21}|\sin\delta_1 - 2|a_{22}||a_{12}|\sin\delta_2\,] ,$$

$$P_{C_{RR}} = \left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$-2|a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1) + 2|a_{12}||a_{21}|\cos\delta_3$$
$$+2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) + 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)$$
$$-2|a_{11}||a_{21}|\sin\delta_1 - 2|a_{22}||a_{12}|\sin\delta_2\,] ,$$

$$P_{C_{LL}} = \left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$-2|a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1) + 2|a_{12}||a_{21}|\cos\delta_3$$
$$-2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) - 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)$$
$$+2|a_{11}||a_{21}|\sin\delta_1 + 2|a_{22}||a_{12}|\sin\delta_2\,] ,$$

and

$$P_{C_{LR}} = \left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$+ 2|a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1) - 2|a_{12}||a_{21}|\cos\delta_3$$
$$+ 2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) + 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)$$
$$+ 2|a_{11}||a_{21}|\sin\delta_1 + 2|a_{22}||a_{12}|\sin\delta_2\,] \,.$$

The right-most subscript in the above expressions for received power refers to the polarization state of the transmitting antenna, while the middle subscript refers to the polarization state of the receiving antenna, both being either right or left circular. Upon adding the first equation to the second, one obtains

$$(26) \qquad P_{C_{RL}} + P_{C_{RR}} = 2\left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$= 2|a_{11}||a_{21}|\sin\delta_1 - 2|a_{22}||a_{12}|\sin\delta_2\,] \,;$$

upon adding the first equation to the third, one obtains

$$(27) \qquad P_{C_{RL}} + P_{C_{LL}} = 2\left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$- 2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) - 2|a_{22}||a_{21}|\sin(\delta_3 - \delta_2)\,] \,;$$

and upon adding the first equation to the fourth, one obtains

$$(28) \qquad P_{C_{RL}} + P_{C_{LR}} = 2\left(\frac{k}{2}\right)^2 |I|^2 [\,|a_{11}|^2 + |a_{22}|^2 + |a_{12}|^2 + |a_{21}|^2$$
$$+ 2|a_{11}||a_{22}|\cos(\delta_3 - \delta_2 - \delta_1) - 2|a_{12}||a_{21}|\cos\delta_3\,] \,.$$

Assuming that $|a_{11}|$, $|a_{22}|$, $|a_{12}|$, $|a_{21}|$, $\cos\delta_1$, and $\cos\delta_2$ have already been determined from power measurements previously described involving only linear antennas, the sign ambiguity can be resolved in δ_1 and δ_2 by using Eq. (26). Then Eq. (27) would result in a choice of two possible values for δ_3, while Eq. (28) would definitely eliminate one of these two possibilities. This scheme might prove simpler and more straightforward for finding the angles than the one involving a combination of linear and circular antennas.

In this section, the complete representation of the transmission, scattering, and reception of a polarized wave is discussed and expressed in matrix form. Then the application of such a representation to the determination of the scattering matrix elements of an unknown surface by measurements with antennas of known characteristics is discussed. A practical method of determination of these elements involving measurement of power or receiving antenna voltage magnitude is explored using linear and circular polarizing antennas. The relationships for power at the receiving antenna for the linear cases are given in Eqs. (19) and (20). Rotation of the receiving antenna is discussed and the power patterns are studied. Ambiguities in signs of the phase differences of the scattering matrix elements determined from power measurements are encountered and resolved simply by employing circular polarizing antennas. The results of this section were derived for surfaces stationary with respect to the antennas. Each element of the scattering matrix is a function of the surface position, angles of incidence and scattering, and frequency of the radiation.

V. INTRODUCTION OF THE STOKES PARAMETERS

Thus far, the various methods of representation of a polarized wave (i.e., linear or Cartesian orthogonal states and the circular orthogonal states) have all had one point in common, i.e., they inherently involved complex elements in the description of the polarized fields, antenna vector heights, and scattering matrix elements. As briefly mentioned, at high frequencies phase difference information is very difficult to measure. On the other hand, intensity or magnitude measurements can generally be made without much difficulty. This is especially true with light waves, where intensity is quite easy to obtain experimentally; but phase differences (e.g., phase difference between the source and the point of reception) are impossible to obtain since many frequencies are present. (This is not to say that the effects of phase differences are not important, since many experiments in light are based upon a shift in phase, such as inteference experiments.) In the previous section, it was mentioned that practical measurements of the scattering matrix elements for an arbitrary surface involved measurements in receiving voltage magnitude or intensity instead of phase differences. All this suggests the possibility of finding another method of representing polarized waves employing pure real elements, since power measurements involve only real physical quantities.

One such method of representation of polarized waves, involving only real elements, is the representation by the Stokes parameters [3, 4, 11]. There are several demands upon such a system involving pure real numbers:

(1) They must contain a complete description of the polarization properties of a wave.

(2) Since it was shown in Section I that any elliptically polarized wave contained essentially three independent pieces of information when it is completely polarized, any new method involving real numbers (even though there may be more than three elements) will contain only three independent elements; therefore there must exist dependency relationships among the remaining elements.

(3) Such a method of representation using pure real numbers must be invariant under a change in the orthogonal representation of the same wave (i.e., as seen previously, any generally elliptically polarized wave can be specified using either linear or circular polarization states; the new method using real numbers must be the same for the same wave, regardless of whether one started from the circular representation or the linear representation).

This latter property of invariance can be dismissed at this point; it has already been indicated (shown in detail for the circular states) that there exists a one-to-one transformation from the many representation states on the Poincare sphere to the Cartesian, or linear states. Since such a transformation is possible, any wave can be specified in linear polarization states and the new system can be developed from these linear states; the implication is that such a representation is the same regardless of whether the wave was actually represented in linear states or not.

At this point, the Stokes parameters for any polarized wave will be developed by proceeding from an established physical quantity, i.e., the vector height of an antenna. Consider a transmitting antenna of arbitrary vector height $[h_t]$ and a receiving antenna of arbitrary vector height $[h_r]$. These vector heights are directly proportional to the polarization of the wave which these antennas would transmit. An expression will be sought for the power at the receiving antenna due to the transmitted wave where there exists no scattering of the transmitted wave. Let the vector heights be specified using the linear method of representation, choosing linear vectors along $\hat{\theta}$ and $\hat{\phi}$ and the angular unit vectors in spherical polar coordinates, and arranging the system so that propagation from transmitter to receiver takes place along a radius from the origin. Except for a constant (which shall be omitted throughout the discussion), the normalized power at the receiver terminals is given by

$$P = \left|[h_r]^T[h_t]\right|^2 = \left| \begin{bmatrix} h_\theta^r & h_\phi^r \end{bmatrix} \begin{bmatrix} h_\theta^t \\ h_\phi^t \end{bmatrix} \right|^2$$

$$= (h_\theta^r h_\theta^t + h_\phi^r h_\phi^t)(h_\theta^{r*} h_\theta^{t*} + h_\phi^{r*} h_\phi^{t*}) .$$

The representation desired is one in which the polarization properties of the transmitting antenna can be represented by pure real numbers; and likewise for the receiving antenna. The power received should be expressible as some matrix product between these sets of real numbers:

$$P = h_\theta^r h_\theta^{r*} h_\theta^t h_\theta^{t*} + h_\phi^r h_\phi^t h_\theta^{r*} h_\theta^{t*} + h_\theta^r h_\theta^t h_\phi^{r*} h_\phi^{t*} + h_\phi^r h_\phi^{r*} h_\phi^t h_\phi^{t*} .$$

If one adds and subtracts the same terms to the above equation and re-arranges them, he can arrive at the following expression:

$$\begin{aligned} P = {} & \tfrac{1}{2} h_\theta^r h_\theta^{r*} h_\theta^t h_\theta^{t*} + \tfrac{1}{2} h_\theta^r h_\theta^{r*} h_\phi^t h_\phi^{t*} + \tfrac{1}{2} h_\phi^r h_\phi^{r*} h_\theta^t h_\theta^{t*} + \tfrac{1}{2} h_\phi^r h_\phi^{r*} h_\phi^t h_\phi^{t*} \\ & + \tfrac{1}{2} h_\theta^r h_\theta^{r*} h_\theta^t h_\theta^{t*} - \tfrac{1}{2} h_\theta^r h_\theta^{r*} h_\phi^t h_\phi^{t*} - \tfrac{1}{2} h_\phi^r h_\phi^{r*} h_\theta^t h_\theta^{t*} + \tfrac{1}{2} h_\phi^r h_\phi^{r*} h_\phi^t h_\phi^{t*} \\ & + \tfrac{1}{2} h_\phi^r h_\theta^{r*} h_\phi^t h_\theta^{t*} + \tfrac{1}{2} h_\phi^{r*} h_\theta^r h_\phi^{t*} h_\theta^t + \tfrac{1}{2} h_\theta^{r*} h_\phi^r h_\theta^{t*} h_\phi^t + \tfrac{1}{2} h_\theta^r h_\phi^{r*} h_\theta^t h_\phi^{t*} \\ & + \tfrac{1}{2} h_\phi^r h_\theta^{r*} h_\phi^t h_\theta^{t*} - \tfrac{1}{2} h_\phi^{r*} h_\theta^r h_\phi^{t*} h_\theta^t - \tfrac{1}{2} h_\theta^{r*} h_\phi^r h_\theta^{t*} h_\phi^t + \tfrac{1}{2} h_\theta^r h_\phi^{r*} h_\theta^t h_\phi^{t*} . \end{aligned}$$

The extra terms were added so that each line of the last expression could be factored as follows:

$$P = \frac{1}{2}(h_\theta^r h_\theta^{r*} + h_\phi^r h_\phi^{r*})(h_\theta^t h_\theta^{t*} + h_\phi^t h_\phi^{t*})$$

$$+\frac{1}{2}(h_\theta^r h_\theta^{r*} - h_\phi^r h_\phi^{r*})(h_\theta^t h_\theta^{t*} - h_\phi^t h_\phi^{t*})$$

$$+\frac{1}{2}(h_\phi^r h_\theta^{r*} + h_\theta^r h_\phi^{r*})(h_\phi^t h_\theta^{t*} + h_\theta^t h_\phi^{t*})$$

$$-\frac{1}{2}(h_\phi^{r*} h_\theta^r - h_\phi^r h_\theta^{r*})(h_\phi^{t*} h_\theta^t - h_\phi^t h_\theta^{t*})\,.$$

In matrix notation, this expression can be written as below:

$$(29) \qquad P = \left[\frac{h_\theta^r h_\theta^{r*} + h_\phi^r h_\phi^{r*}}{\sqrt{2}},\ \frac{h_\theta^r h_\theta^{r*} - h_\phi^r h_\phi^{r*}}{\sqrt{2}},\ \frac{h_\phi^r h_\theta^{r*} + h_\theta^r h_\phi^{r*}}{\sqrt{2}},\ \frac{h_\phi^r h_\theta^{r*} - h_\theta^r h_\phi^{r*}}{j\sqrt{2}} \right] \begin{bmatrix} \dfrac{h_\theta^t h_\theta^{t*} + h_\phi^t h_\phi^{t*}}{\sqrt{2}} \\ \dfrac{h_\theta^t h_\theta^{t*} - h_\phi^t h_\phi^{t*}}{\sqrt{2}} \\ \dfrac{h_\phi^t h_\theta^{t*} + h_\theta^t h_\phi^{t*}}{\sqrt{2}} \\ \dfrac{h_\phi^t h_\theta^{t*} - h_\theta^t h_\phi^{t*}}{j\sqrt{2}} \end{bmatrix}$$

or

$$(30) \qquad P = \left[\frac{|h_\theta^r|^2 + |h_\phi^r|^2}{\sqrt{2}},\ \frac{|h_\theta^r|^2 - |h_\phi^r|^2}{\sqrt{2}},\ \frac{2\mathrm{Re}\{h_\phi^r h_\theta^{r*}\}}{\sqrt{2}},\ \frac{2\mathrm{Im}\{h_\phi^r h_\theta^{r*}\}}{\sqrt{2}} \right] \begin{bmatrix} \dfrac{|h_\theta^t|^2 + |h_\phi^t|^2}{\sqrt{2}} \\ \dfrac{|h_\theta^t|^2 - |h_\phi^t|^2}{\sqrt{2}} \\ \dfrac{2\mathrm{Re}\{h_\phi^t h_\theta^{t*}\}}{\sqrt{2}} \\ \dfrac{2\mathrm{Im}\{h_\phi^t h_\theta^{t*}\}}{\sqrt{2}} \end{bmatrix}$$

At this point, it is convenient to re-define the above matrix elements as Stokes parameters:

(31) $$h_1^r \equiv \frac{|h_\theta^r|^2 + |h_\phi^r|^2}{\sqrt{2}} = \frac{h_\theta^r h_\theta^{r*} + h_\phi^r h_\phi^{r*}}{\sqrt{2}} \quad ;$$

$$h_1^t \equiv \frac{|h_\theta^t|^2 + |h_\phi^t|^2}{\sqrt{2}} = \frac{h_\theta^t h_\theta^{t*} + h_\phi^t h_\phi^{t*}}{\sqrt{2}} \quad ,$$

$$h_2^r \equiv \frac{|h_\theta^r|^2 - |h_\phi^r|^2}{\sqrt{2}} = \frac{h_\theta^r h_\theta^{r*} - h_\phi^r h_\phi^{r*}}{\sqrt{2}} \quad ;$$

$$h_2^t \equiv \frac{|h_\theta^t|^2 - |h_\phi^t|^2}{\sqrt{2}} = \frac{h_\theta^t h_\theta^{t*} - h_\phi^t h_\phi^{t*}}{\sqrt{2}} \quad ,$$

$$h_3^r \equiv \frac{2\mathrm{Re}\{h_\phi^r h_\theta^{r*}\}}{\sqrt{2}} = \frac{h_\phi^r h_\theta^{r*} + h_\theta^r h_\phi^{r*}}{\sqrt{2}} \quad ;$$

$$h_3^t \equiv \frac{2\mathrm{Re}\{h_\phi^t h_\theta^{t*}\}}{\sqrt{2}} = \frac{h_\phi^t h_\theta^{t*} + h_\theta^t h_\phi^{t*}}{\sqrt{2}} \quad ,$$

and

$$h_4^r \equiv \frac{2\mathrm{Im}\{h_\phi^r h_\theta^{r*}\}}{\sqrt{2}} = \frac{h_\phi^r h_\theta^{r*} - h_\theta^r h_\phi^{r*}}{j\sqrt{2}} \quad ;$$

$$h_4^t \equiv \frac{2\mathrm{Im}\{h_\phi^t h_\theta^{t*}\}}{\sqrt{2}} = \frac{h_\phi^t h_\theta^{t*} - h_\theta^t h_\phi^{t*}}{j\sqrt{2}} \quad .$$

Therefore, the power received may be expressed in terms of these newly defined Stokes parameters:

$$(32) \qquad P = \begin{bmatrix} h_1^r & h_2^r & h_3^r & h_4^r \end{bmatrix} \begin{bmatrix} h_1^t \\ h_2^t \\ h_3^t \\ h_4^t \end{bmatrix} .$$

These Stokes parameters have the following properties, evident from the above expressions:

(1) The elements have been separated into two matrices; each represents the polarization properties of the wave which would be transmitted from that antenna. Each element of a matrix is given in terms of combinations of the components of the vector height of that antenna.

(2) The product of two such matrices represents received power, whereas the product of two vector height matrices represents a complex received voltage.

(3) Each Stokes parameter is pure real; this is easily seen from the form of the first definition of these quantities in Eq. (31).

(4) There are four elements in each matrix representing the polarization state of a wave transmitted from that antenna; since there are only three independent pieces of information present in any completely polarized wave, these four elements are not independent. There exists one dependency relationship between the Stokes parameters.

In order to discover the dependency relationship between the Stokes parameters and gain a physical interpretation of this dependency, assume that the receiving antenna is identical with the transmitting antenna and oriented so that it receives all the power possible from the transmitter (i. e.,

$$h_\theta^r = h_\theta^t , \; h_\phi^r = h_\phi^t ; \; \therefore h_1^r = h_1^t , \; h_2^r = h_2^t , \; h_3^r = h_3^t , \; h_4^r = h_4^t).$$

From basic considerations, the power received in such a case is

$$P = |[h^r]^T[h^t]|^2 = \left| \begin{bmatrix} h_\theta^t h_\phi^t \end{bmatrix} \begin{bmatrix} h_\theta^t \\ h_\phi^t \end{bmatrix} \right|^2$$

$$= |h_\theta^t h_\theta^t + h_\phi^t h_\phi^t|^2 = (|h_\theta^t|^2 + |h_\phi^t|^2)^2 = 2(h_1^t)^2 \quad .$$

The right-most side of this equation is evident from the definition of h_1^t. Now, start with Eq. (32) and find the power received in this same case when using the Stokes parameters:

$$P = \begin{bmatrix} h_1^t & h_2^t & h_3^t & h_4^t \end{bmatrix} \begin{bmatrix} h_1^t \\ h_2^t \\ h_3^t \\ h_4^t \end{bmatrix} = (h_1^t)^2 + (h_2^t)^2 + (h_3^t)^2 + (h_4^t)^2 \quad .$$

Equating these two expressions for power received yields the following relationship:

$$(33) \qquad 2(h_1^t)^2 = (h_1^t)^2 + (h_2^t)^2 + (h_3^t)^2 + (h_4^t)^2 \quad ,$$

$$\therefore (h_1^t)^2 = (h_2^t)^2 + (h_3^t)^2 + (h_4^t)^2 \quad .$$

This above relationship is valid not only for the transmitting antenna, but for any completely polarized wave represented by the Stokes parameters. This result may be easily verified in general by substituting into it the definitions of the Stokes parameters, Eq. (31), and carrying out the indicated algebra.

In this section, a physical motivation for the definition of pure real elements to represent a polarized wave (the Stokes parameters) was first provided. Then these parameters were defined as matrix elements related to the vector height of (or wave transmitted by) an antenna. These parameters were defined in this manner so that it would be obvious that the entire process of transmission and reception of the power of a wave can be represented by matrix multiplication of these pure real Stokes parameters, just as the transmission and reception of tne complex voltage of a wave can be represented by matrix multiplication of the complex vector height of the antennas. The definitions of the Stokes parameters thus developed are given in Eq. (31), and the matrix multiplication representing transmission and reception of power are given in Eq. (32). Since there were four real Stokes parameters (instead of three) representing the polarization of a wave, it was noted that there existed one dependency relationship between the parameters; this was developed and is given in Eq. (33).

VI. THE SCATTERING MATRIX FOR THE STOKES PARAMETERS

In the previous section it was shown that the expression for the power received by an antenna coming from an elliptically polarized wave propagating directly from the transmitting antenna could be represented in two forms; one employs the vector height matrices introduced in Section III and the other employs the newly defined Stokes parameter matrices, which are themselves functions of the vector heights. The two representations are repeated below:

$$P = k\left|[h_r]^T[h_t]\right|^2 ;$$

$[h_r]$ and $[h_t]$ are two-by-two complex matrices, vector heights of transmitting and receiving antennas; and

$$P = k[H_r]^T[H_t] ;$$

$[H_r]$ and $[H_t]$ are four-by-one real matrices, the Stokes parameters of the vector heights of transmitting and receiving antenna, defined by Eq. (31).

It is evident that in most instances where received power is the variable of interest, the second representation employing the Stokes parameters is more simple and straightforward to use. Consisting of only real elements, this representation is readily adaptable to experimentation involving real physical observables.

It was shown in Section IV that the process of transmission, scattering by a surface, and reception also has a matrix representation in terms of the vector heights and the two-by-two complex scattering matrix. The power received after such a process is immediately evident as the following:

(34) $$P = k\left| [h_r]^T [a][h_t] \right|^2 \; ;$$

here the elements of $[h_r]$, $[a]$, and $[h_t]$ must be specified in the same polarization representation; i.e., all linear, all circular, etc. The elements of $[a]$ are defined in Section II.

For the same reasons mentioned previously, such a representation as that above is cumbersome; and since the matrices involve complex elements, this representation is not easily amenable to the interpretation of experimental data, as seen in Section IV.

It would seem a simple extension to define a pure real, four-by-four matrix relating uniquely the Stokes parameters of the scattered wave from the surface to the Stokes parameters of the wave incident upon the surface. This would yield the following, much simpler form for the power received after such a process:

(35) $$P = k[H_r]^T[A][H_t] \; .$$

The elements of $[A]$ are yet to be determined and defined. However, several properties of these elements can be stated beforehand. One would expect and require that the elements of $[A]$ should be functions only of the elements of $[a]$, just as the elements of $[H_r]$ are functions only of the elements of $[h_r]$, and the elements of $[H_t]$ are functions only of $[h_t]$. Another way of saying the same thing is to state that in order to be useful, the elements of $[A]$ should be functions of the surface only and invariant of the form of the incident radiation. This requirement will be met in the development of $[A]$. Since the original scattering matrix $[a]$ contained seven independent quantities in general, one would expect that only seven of the elements of $[A]$ would be independent, or, in other words, the matrix $[A]$ should contain combinations of only seven independent quantities. This must be true since both $[A]$ and $[a]$ completely characterize the polarization properties of the scattered wave in terms of those of the incident wave. Since there are 16 real elements in $[A]$, there must therefore exist nine dependency relationships between these elements. Note that in the case of backscattering it was shown that $a_{12} = a_{21}$ in the matrix $[a]$. This means that the number of independent quantities in $[a]$ is reduced to five instead of seven for this special case. Thus, only five of the elements of $[A]$ should be independent in the case of backscattering.

The following procedure will be followed in the development of the scattering matrix [A]: (1) the Stokes parameters of an incident and scattered wave will be defined in terms of the properties of the polarized incident and scattered electric field; (2) the scattered electric field will then be expressed in terms of the incident electric field and the elements of the scattering matrix [a]; and (3) the Stokes parameters of the scattered wave will then be found in terms of the Stokes parameters of the incident wave and various combinations of the elements of [a]. This will then define the elements of [A].

The Stokes parameters of the incident field are H_1^i, H_2^i, H_3^i, H_4^i; the Stokes parameters of the scattered field are H_1^s, H_2^s, H_3^s, H_4^s. The matrix [A] expressing the relationship between them for a given surface appears in the following equation:

$$\begin{bmatrix} H_1^s \\ H_2^s \\ H_3^s \\ H_4^s \end{bmatrix} = \frac{1}{4\pi r_s^2} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix} \begin{bmatrix} H_1^i \\ H_2^i \\ H_3^i \\ H_4^i \end{bmatrix} \quad ;$$

r_s = distance from the surface to the point of observation of the scattered wave.

The Stokes parameters of the incident and scattered field are defined as follows, using Eq. (31):

$$(37) \qquad H_1^s \equiv \frac{E_\theta^s E_\theta^{s*} + E_\phi^s E_\phi^{s*}}{\sqrt{2}} \quad ; \quad H_1^i \equiv \frac{E_\theta^i E_\theta^{i*} + E_\phi^i E_\phi^{i*}}{\sqrt{2}} \quad ,$$

$$H_2^s \equiv \frac{E_\theta^s E_\theta^{s*} - E_\phi^s E_\phi^{s*}}{\sqrt{2}} \quad ; \quad H_2^i \equiv \frac{E_\theta^i E_\theta^{i*} - E_\phi^i E_\phi^{i*}}{\sqrt{2}} \quad ,$$

$$H_3^s \equiv \frac{E_\phi^s E_\theta^{s*} + E_\theta^s E_\phi^{s*}}{\sqrt{2}} \quad ; \quad H_3^i \quad \frac{E_\phi^i E_\theta^{i*} + E_\theta^i E_\phi^{i*}}{\sqrt{2}} \quad ,$$

and

$$H_4{}^s \equiv \frac{E_\phi^s E_\theta^{s*} - E_\theta^s E_\phi^{s*}}{j\sqrt{2}} \quad ; \; H_4{}^s \equiv \frac{E_\phi^i E_\theta^{i*} - E_\theta^i E_\phi^{i*}}{j\sqrt{2}} \quad .$$

The elementary scattering matrix [a] relating E_θ^s and E_ϕ^s to E_θ^i and E_ϕ^i is repeated below, along with its complex conjugate relationship:

$$\text{(38)} \qquad \begin{bmatrix} E_\theta^s \\ E_\phi^s \end{bmatrix} = \frac{1}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_\theta^i \\ E_\phi^i \end{bmatrix} \quad ;$$

$$\begin{bmatrix} E_\theta^{s*} \\ E_\phi^{s*} \end{bmatrix} = \frac{1}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{11}^* & a_{12}^* \\ a_{21}^* & a_{22}^* \end{bmatrix} \begin{bmatrix} E_\theta^{i*} \\ E_\phi^{i*} \end{bmatrix} \quad .$$

In the definition (Eq. (37)) of the Stokes parameters of the scattered wave, the following quantities appear;

$$E_\theta^s E_\theta^{s*}, \; E_\phi^s E_\theta^{s*}, \; E_\theta^s E_\phi^{s*}, \text{ and } E_\phi^s E_\phi^{s*}.$$

These quantities can be easily determined for Eq. (38), in terms of the incident field and the elements of the scattering matrix. Their relationship is best summarized by the following matrix:

$$\text{(39)} \qquad \begin{bmatrix} E_\theta^s E_\theta^{s*} \\ E_\phi^s E_\theta^{s*} \\ E_\theta^s E_\phi^{s*} \\ E_\phi^s E_\phi^{s*} \end{bmatrix} = \frac{1}{\sqrt{4\pi r_s^2}} \begin{bmatrix} a_{11} a_{11}^* & a_{12} a_{11}^* & a_{11} a_{12}^* & a_{12} a_{12}^* \\ a_{21} a_{11}^* & a_{22} a_{11}^* & a_{21} a_{12}^* & a_{22} a_{12}^* \\ a_{11} a_{21}^* & a_{12} a_{21}^* & a_{11} a_{22}^* & a_{12} a_{22}^* \\ a_{21} a_{21}^* & a_{22} a_{21}^* & a_{21} a_{22}^* & a_{22} a_{22}^* \end{bmatrix} \begin{bmatrix} E_\theta^i E_\theta^{i*} \\ E_\phi^i E_\theta^{i*} \\ E_\theta^i E_\phi^{i*} \\ E_\phi^i E_\phi^{i*} \end{bmatrix} \quad .$$

Now the quantities in Eq. (39) will be substituted into the definitions of the Stokes parameters of the scattered field. Consider first H_1^s .

$$\sqrt{2}\, H_1^s = E_\theta^s E_\theta^{s*} + E_\phi^s E_\phi^{s*} = a_{11} a_{11}^* \, E_\theta^i E_\theta^{i*} + a_{12} a_{11}^* \, E_\phi^i E_\theta^{i*}$$

$$+ a_{11} a_{12}^* \, E_\theta^i E_\phi^{i*} + a_{12} a_{12}^* \, E_\phi^i E_\phi^{i*} + a_{21} a_{21}^* \, E_\theta^i E_\theta^{i*}$$

$$+ a_{22} a_{21}^* \, E_\phi^i E_\theta^{i*} + a_{21} a_{22}^* \, E_\theta^i E_\phi^{i*} + a_{22} a_{22}^* \, E_\phi^i E_\phi^{i*}$$

or

$$\sqrt{2}\, H_1^s = k_{11} E_\theta^i E_\theta^{i*} + k_{12} E_\phi^i E_\theta^{i*} + k_{13} E_\theta^i E_\phi^{i*} + k_{14} E_\phi^i E_\phi^{i*} ,$$

where

$$k_{11} = (a_{11} a_{11}^* + a_{21} a_{21}^*) \quad ; \quad k_{12} = (a_{12} a_{11}^* + a_{22} a_{21}^*)$$

$$k_{13} = (a_{11} a_{12}^* + a_{21} a_{22}^*) \quad ; \quad k_{14} = (a_{12} a_{12}^* + a_{22} a_{22}^*) \quad .$$

Now $\sqrt{2}\, H_1^s$ can be re-arranged as follows:

$$\sqrt{2}\, H_1^s = \frac{1}{2}(k_{11} + k_{14})\, E_\theta^i E_\theta^{i*} + \frac{1}{2}(k_{11} + k_{14})\, E_\phi^i E_\phi^{i*}$$

$$+ \frac{1}{2}(k_{11} - k_{14})\, E_\theta^i E_\theta^{i*} + \frac{1}{2}(k_{11} - k_{14})\, E_\phi^i E_\phi^{i*}$$

$$+ \frac{1}{2}(k_{12} + k_{13})\, E_\phi^i E_\theta^{i*} + \frac{1}{2}(k_{12} + k_{13})\, E_\theta^i E_\phi^{i*}$$

$$+ \frac{1}{2} j(k_{12} - k_{13}) \frac{E_\phi^i E_\theta^{i*}}{j} - \frac{1}{2} j(k_{12} - k_{13}) \frac{E_\theta^i E_\phi^{i*}}{j} \quad .$$

When written in the above form the Stokes parameters of the incident wave are obvious, i.e.,

$$2H_1^s = (k_{11} + k_{14})H_1^i + (k_{11} - k_{14})H_2^i + (k_{12} + k_{13})H_3^i + j(k_{12} - k_{13})H_4^i .$$

From this, the following elements of the scattering matrix [A] are obvious: α_{11}, α_{12}, α_{13}, α_{14}. They are written in two alternative forms:

the second form employs the definitions of the phase differences between the elements of the scattering matrix [a] as defined in Section IV by Eq. (21).

$$\alpha_{11} = \frac{1}{2}(k_{11}+k_{14}) = \frac{1}{2}(a_{11}a_{11}^* + a_{21}a_{21}^* + a_{12}a_{12}^* + a_{22}a_{22}^*) \tag{40}$$

$$= \frac{1}{2}[\,|a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2\,] ,$$

and

$$\alpha_{12} = \frac{1}{2}(k_{11} - k_{14}) = \frac{1}{2}(a_{11}a_{11}^* + a_{21}a_{21}^* - a_{12}a_{12}^* - a_{22}a_{22}^*)$$

$$= \frac{1}{2}[\,|a_{11}|^2 + |a_{21}|^2 - |a_{12}|^2 - |a_{22}|^2\,] ,$$

$$\alpha_{13} = \frac{1}{2}(k_{12} + k_{13}) = \frac{1}{2}(a_{12}a_{11}^* + a_{22}a_{21}^* + a_{11}a_{12}^* + a_{21}a_{22}^*)$$

$$= \frac{1}{2}[2|a_{11}||a_{12}|\cos(\delta_3 - \delta_1) + 2|a_{21}||a_{22}|\cos(\delta_3 - \delta_2)] ,$$

and

$$\alpha_{14} = \frac{j}{2}(k_{12} - k_{13}) = \frac{j}{2}(a_{12}a_{11}^* + a_{22}a_{21}^* - a_{11}a_{12}^* - a_{21}a_{22}^*)$$

$$= \frac{1}{2}[2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) + 2|a_{21}||a_{22}|\sin(\delta_3 - \delta_2)] .$$

The other elements of the matrix [A] will be derived in exactly the same manner.

$$\sqrt{2}\, H_2{}^s = E_\theta^s E_\theta^{s*} - E_\phi^s E_\phi^{s*} = a_{11}a_{11}{}^* E_\theta^i E_\theta^{i*} + a_{12}a_{11}{}^* E_\phi^i E_\theta^{i*}$$
$$+ a_{11}a_{12}{}^* E_\theta^i E_\phi^{i*} + a_{12}a_{12}{}^* E_\phi^i E_\phi^{i*} - a_{21}a_{21}{}^* E_\theta^i E_\theta^{i*}$$
$$- a_{22}a_{21}{}^* E_\phi^i E_\theta^{i*} - a_{21}a_{22}{}^* E_\theta^i E_\phi^{i*} - a_{22}a_{22}{}^* E_\phi^i E_\phi^{i*} .$$

In exactly the same manner as before, define

(41) cont.

$$k_{21} = a_{11} a_{11}^{*} - a_{21} a_{21}^{*} \quad ; \quad k_{22} = a_{12} a_{11}^{*} - a_{22} a_{21}^{*}$$

$$k_{23} = a_{11} a_{12}^{*} - a_{21} a_{22}^{*} \quad ; \quad k_{24} = a_{12} a_{12}^{*} - a_{22} a_{22}^{*} .$$

$$\therefore \quad 2H_2^{s} = (k_{21} + k_{24})H_1^{i} + (k_{21} - k_{24})H_2^{i} + (k_{22} + k_{23})H_3^{i} + j(k_{22} - k_{22})H_4^{i};$$

$$\therefore \quad \alpha_{21} = \frac{1}{2}(k_{21} + k_{24}) = \frac{1}{2}(a_{11} a_{11}^{*} - a_{21} a_{21}^{*} + a_{12} a_{12}^{*} - a_{22} a_{22}^{*})$$

$$= \frac{1}{2}[\,|a_{11}|^2 - |a_{21}|^2 + |a_{12}|^2 - |a_{22}|^2] ;$$

$$\alpha_{22} = \frac{1}{2}(k_{21} - k_{24}) = \frac{1}{2}(a_{11} a_{11}^{*} - a_{21} a_{21}^{*} - a_{12} a_{12}^{*} + a_{22} a_{22}^{*})$$

$$= \frac{1}{2}[\,|a_{11}|^2 - |a_{21}|^2 - |a_{12}|^2 + |a_{22}|^2] ,$$

$$\alpha_{23} = \frac{1}{2}(k_{22} + k_{23}) = \frac{1}{2}(a_{12} a_{11}^{*} - a_{22} a_{21}^{*} + a_{11} a_{12}^{*} - a_{21} a_{22}^{*})$$

$$= \frac{1}{2}[2|a_{11}||a_{12}|\cos(\delta_3 - \delta_1) - 2|a_{21}||a_{22}|\cos(\delta_3 - \delta_2)] ,$$

and

$$\alpha_{24} = \frac{j}{2}(k_{22} - k_{23}) = \frac{j}{\sqrt{2}}(a_{12} a_{11}^{*} - a_{22} a_{21}^{*} - a_{11} a_{12}^{*} + a_{21} a_{22}^{*})$$

$$= \frac{1}{2}[2|a_{11}||a_{12}|\sin(\delta_3 - \delta_1) - 2|a_{21}||a_{22}|\sin(\delta_3 - \delta_2)] .$$

$$\sqrt{2}\, H_3^{s} = E_\phi^s E_\theta^{s*} + E_\theta^s E_\phi^{s*} = a_{21} a_{11}^{*} E_\theta^i E_\theta^{i*} + a_{22} a_{11}^{*} E_\phi^i E_\theta^{i*}$$

$$+ a_{21} a_{12}^{*} E_\theta^i E_\phi^{i*} + a_{22} a_{12}^{*} E_\phi^i E_\phi^{i*} + a_{11} a_{21}^{*} E_\theta^i E_\theta^{i*}$$

$$+ a_{12} a_{21}^{*} E_\phi^i E_\theta^{i*} + a_{11} a_{22}^{*} E_\theta^i E_\phi^{i*} + a_{12} a_{22}^{*} E_\phi^i E_\phi^{i*} .$$

Define

(42)

$$k_{31} = a_{21} a_{11}^{*} + a_{11} a_{21}^{*} \quad ; \quad k_{32} = a_{22} a_{11}^{*} \; a_{12} a_{21}^{*}$$

$$k_{33} = a_{21} a_{12}^{*} + a_{11} a_{22}^{*} \quad ; \quad k_{34} = a_{22} a_{12}^{*} + a_{12} a_{22}^{*} .$$

(42) cont. $\therefore \quad 2H_3^s = (k_{31}+k_{34})H_1^i + (k_{31}-k_{34})H_2^i + (k_{32}+k_{33})H_3^i + j(k_{32}-k_{33})H_4^i;$

$$\therefore \quad \alpha_{31} = \frac{1}{2}(k_{31}+k_{34}) = \frac{1}{2}(a_{21}a_{11}^* + a_{11}a_{21}^* + a_{22}a_{12}^* + a_{11}a_{22}^*)$$

$$= \frac{1}{2}\left[2|a_{11}||a_{21}|\cos\delta_1 + 2|a_{12}||a_{22}|\cos\delta_2\right];$$

$$\alpha_{32} = \frac{1}{2}(k_{31}-k_{34}) = \frac{1}{2}(a_{21}a_{11}^* + a_{11}a_{21}^* - a_{22}a_{12}^* - a_{11}a_{22}^*)$$

$$= \frac{1}{2}\left[2|a_{11}||a_{21}|\cos\delta_1 - 2|a_{12}||a_{22}|\cos\delta_2\right],$$

$$\alpha_{33} = \frac{1}{2}(k_{32}+k_{33}) = \frac{1}{2}(a_{22}a_{11}^* + a_{12}a_{21}^* + a_{21}a_{12}^* + a_{11}a_{22}^*)$$

$$= \frac{1}{2}\left[2|a_{11}||a_{22}|\cos(\delta_3-\delta_2-\delta_1) + 2|a_{12}||a_{21}|\cos\delta_3\right],$$

and

$$\alpha_{34} = \frac{j}{2}(k_{32}-k_{33}) = \frac{j}{2}(a_{22}a_{11}^* + a_{12}a_{21}^* - a_{21}a_{12}^* - a_{11}a_{22}^*)$$

$$= \frac{1}{2}\left[2|a_{11}||a_{22}|\sin(\delta_3-\delta_2-\delta_1) + 2|a_{12}||a_{21}|\sin\delta_3\right].$$

$$j\sqrt{2}\,H_4^s = E_\phi^s E_\theta^{s*} - E_\theta^s E_\phi^{s*} = a_{21}a_{11}^* E_\theta^i E_\theta^{i*} + a_{22}a_{11}^* E_\phi^i E_\theta^{i*}$$
$$+ a_{21}a_{12}^* E_\theta^i E_\phi^{i*} + a_{22}a_{12}^* E_\phi^i E_\phi^{i*} - a_{11}a_{21}^* E_\theta^i E_\theta^{i*}$$
$$- a_{12}a_{21}^* E_\phi^i E_\theta^{i*} - a_{11}a_{22}^* E_\theta^i E_\phi^{i*} - a_{12}a_{22}^* E_\phi^i E_\phi^{i*}.$$

Define

(43) $k_{41} = a_{21}a_{11}^* - a_{11}a_{21}^* \quad ; \quad k_{42} = a_{22}a_{11}^* - a_{12}a_{21}^*$

$k_{43} = a_{21}a_{12}^* - a_{11}a_{22}^* \quad ; \quad k_{44} = a_{22}a_{12}^* - a_{12}a_{22}^*.$

$$\therefore \quad 2H_4^s = -j(k_{41}+k_{44})H_1^i - j(k_{41}-k_{44})H_2^i - j(k_{42}+k_{43})H_3^i + (k_{42}-k_{43})H_4^i ;$$

$$\therefore \quad \alpha_{41} = \frac{-j}{2}(k_{41}+k_{44}) = \frac{j}{2}(a_{11}a_{21}^* - a_{21}a_{11}^* + a_{12}a_{22}^* - a_{22}a_{12}^*)$$

$$= \frac{1}{2}[2|a_{11}||a_{21}|\sin\delta_1 + 2|a_{12}||a_{22}|\sin\delta_2] ;$$

$$\alpha_{42} = \frac{-j}{2}(k_{41}-k_{44}) = \frac{j}{2}(a_{11}a_{21}^* - a_{21}a_{11}^* - a_{12}a_{22}^* + a_{22}a_{12}^*)$$

$$= \frac{1}{2}[2|a_{11}||a_{21}|\sin\delta_1 - 2|a_{12}||a_{22}|\sin\delta_2] ,$$

$$\alpha_{43} = \frac{-j}{2}(k_{42}+k_{43}) = \frac{j}{2}(a_{12}a_{21}^* - a_{22}a_{11}^* + a_{11}a_{22}^* - a_{21}a_{12}^*)$$

$$= \frac{1}{2}[-2|a_{11}||a_{22}|\sin(\delta_3-\delta_2-\delta_1) + 2|a_{12}||a_{21}|\sin\delta_3] ,$$

and

$$\alpha_{44} = \frac{1}{2}(k_{42}-k_{43}) = \frac{1}{2}(a_{22}a_{11}^* - a_{11}a_{21}^* - a_{21}a_{12}^* + a_{11}a_{22}^*)$$

$$= \frac{1}{2}[2|a_{11}||a_{22}|\cos(\delta_3-\delta_2-\delta_1) - 2|a_{12}||a_{21}|\cos\delta_3] .$$

Thus Eqs. (35) - (38) constitute the definitions of the 16 real elements of the scattering matrix relating the Stokes parameters of the scattered field to the Stokes parameters of the incident field. These elements are defined strictly in terms of the elements of the amplitude scattering matrix [a]. These elements, as seen above, can be expressed entirely in terms of the four amplitudes and three phase differences of the elements in [a]. This shows that there are only seven independent quantities in the Stokes scattering matrix [A].

Notice that this matrix [A] is <u>not</u> symmetrical, in general; this matrix is not symmetrical even in the special case of back-scattering, when $a_{12} = a_{21}$, as can be seen by comparing α_{41} to α_{14}.

The developments of this section suggest that in many cases it might be easier to determine all the real elements of the Stokes scattering matrix [A] from measurement, knowing the Stokes parameters of the vector heights of the receiving and transmitting antennas (i. e., H_r^1, H_r^2, H_r^3, H_r^4, and H_t^1, H_t^2, H_t^3, H_t^4) by using Eq. (30). Certainly knowing all the elements of the Stokes scattering matrix [A] also determines uniquely all the elements of the elementary scattering matrix [a]; as a matter of fact not all the 16 elements of [A] need be known in order to determine the seven independent quantities in [a]. Only seven need be known, but they must be the correct seven. In many cases it might be easier to proceed in this manner than in the manner of Section IV, which attempted to find the seven quantities of [a] directly by power measurement.

In this section the need for a four-by-four real matrix relating the Stokes parameters of the field scattered from a surface to the Stokes parameters of the incident field was shown and the properties of such a matrix were discussed. The role of such a matrix in the measurement of power received by an antenna after being transmitted and scattered by a surface is shown in Eq. (30). The actual elements of this matrix are developed in terms of the elements of the original scattering matrix [a], and the relations between them are given in Eqs. (35) - (38).

REFERENCES

1. "Effects of the Type of Polarization on Echo Characteristics," Reports 389-9, 16 June 1951 and 389-17, 16 December 1952, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract Number AF 28(099)-90, Rome Air Development Center, Griffiss Air Force Base, N. Y. (ATI-159608) and (AD-21812).

2. Sinclair, G., "Modification of the Radar Range Equation for Arbitrary Targets and Arbitrary Polarization, Report 302-19, 25 September 1948, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract Number W-36-039 sc 33634, Signal Corps Supply Agency, Laboratory Procurement Office, Fort Monmouth, New Jersey, (ATI-44490).

3. Chandrasekhar, S., Radiative Transfer, Oxford University Press, 1950.

4. VanDer Hulst, H. C., Light Scattering by Small Particles, John Wiley and Sons, Inc., (1957).

5. Ko, H. C., "On the Response of a Radio Antenna to Complex Radio Waves," Proceedings of National Electronics Conference, Vol. 17, pp. 500-508, October, 1961.

6. Copeland, J. R., "Radar Target Classification by Polarization Properties," Proceedings of the IRE, Vol. 48, No. 7, July, 1960.

7. Stratton, J., Electromagnetic Theory, McGraw-Hill Book Co., Inc., New York, N. Y., 1941.

8. Ko, H. C., "On the Reception of Quasi-Monochromatic, Partially Polarized Radio Waves," Proceedings of the IRE, Vol. 50, No. 9, September, 1962.

9. Kouyoumjian, R., "Some Special Notes on the Polarization of Electromagnetic Waves, 22 June 1963. (unpublished).

10. Rumsey, et al. "Techniques for Handling Elliptically Polarized Waves with Special Reference to Antennas," Proceedings of IRE, 30, May, 1951.

11. Born, M., and Wolf, E., Principles of Optics, Pergamon Press, 1959, 2nd Ed. 1964.

12. Kennaugh, E.M., "Polarization Properties of Radar Reflections," Report 389-12, 1 March 1952, Antenna Laboratory, The Ohio State University Research Foundation; prepared under Contract Number AF 28(099)-90, Rome Air Development Center, Griffiss Air Force Base, New York, (AD 2494).

13. Silver, J., Microwave Antenna Theory and Design, McGraw-Hill Book Co., Inc. New York, 1949.

14. Schelkunoff, S. A., Electromagnetic Waves, D. Van Nostrand Company, 1943.